The Induction Principles

As always, I'm defining "natural number" to mean the set of integers greater than or equal to 0, i.e. $\{0, 1, 2, \ldots\}$. The first principle of induction is the weak principle of induction:

Weak principle of induction: Suppose we have some propositions $P(n)$ for each natural number n. Suppose that:

- *Base case:* $P(0)$ is true, and
- *Induction step:* for each natural number k, if $P(k)$ true then so is $P(k+1)$.

Then $P(n)$ is true for each natural number n.

Compare this to the strong principle of induction:

Strong principle of induction: Suppose we have some propositions $P(n)$ for each natural number n. Suppose that:

- *Base case:* $P(0)$ is true, and
- *Induction step:* for each natural number k, if $P(0), P(1), \cdots, P(k)$ are true then so is $P(k+1)$.

Then $P(n)$ is true for each natural number n.

The only difference is in the induction step: when you're doing strong induction, you get to assume $P(0)$ through $P(k)$ are true when proving $P(k+1)$. If you're using weak inudction, then you only get to assume $P(k)$ is true when proving $P(k+1)$. Thus, it's *easier* to prove things using strong induction: you have more information to work with when you're trying to prove $P(k+1)$! It's called the "strong" principal of induction because it's (at first glance) a stronger result, meaning you can use it to prove more things.

Different base cases

What if we want to prove that a statement is true for all natural numbers ≥ 1? Or all natural numbers $\geq a$ for some $a \in \mathbb{N}$? Then we can use this modified principle of induction:

Lemma 0.1 (Weak induction with a shift)**.** *Let $a \in \mathbb{N}$, and suppose we have some propositions $P'(n)$ for each natural number $n \geq a$. Suppose that*

- Modified base case: $P'(a)$ *is true, and*
- Modified induction step: *for each natural number $k \geq a$, if $P(k)$ true then so is $P(k+1)$.*

Then $P(n)$ is true for each natural number $n \geq a$.

Proof. For each $m \in \mathbb{N}$ define $P(m)$ to be the proposition $P'(m+a)$. Then, by the modified base case, $P(0) = P'(a)$ is true. Now, let $k \in \mathbb{N}$ be arbitrary, and suppose that $P(k)$ is true. Then, by definition, so is $P'(k+a)$. Since $k \geq 0$, we have $k+a \geq a$, so by the modified induction step we know $P'(k+a+1)$ is true. But $P'(k+a+1) = P(k+1)$, so $P(k+1)$ is true. By our original principle of induction (starting at 0), we have $P(m)$ is true for all $m \in \mathbb{N}$. By definition, this means $P'(n)$ is true for all $n \geq a$. □

So, if you're faced with math problem in the wild, how do you know what to use as your base case? It depends on what you're trying to prove! The above lemma says that if you want to prove $P(n)$ is true for all $n \geq a$, you can use $n = a$ as your base case.

Optional exercise: state and prove a shifted version of the strong principal of induction.