Math 4400 Homework 3
Due: Monday, June 5th, 2017

Feel free to work with your classmates, but everyone must turn in their own assignment. Please make a note of who you worked with on each problem. Let me know if you find a typo, or you're stuck on any of the problems.

1. (10 points) Suppose a and b are nonzero integers. Suppose also that $a \mid b$ and $b \mid a$. Prove (carefully!) that $a = \pm b$.

2. (10 points) Recall that, by definition, we say $a \equiv b \mod n$ if $n \mid (a - b)$. Now, let $x, y, z, n \in \mathbb{Z}$ with $n > 0$. Prove the following facts:
   (a) $x \equiv x \mod n$
   (b) If $x \equiv y \mod n$, then $y \equiv x \mod n$
   (c) If $x \equiv y \mod n$ and $y \equiv z \mod n$, then $x \equiv z \mod n$.

   The fact that these three statements hold means that "$\_ \equiv \_ \mod n$" is an *equivalence relation*.

3. (a) (10 points) Suppose that $ac \equiv bc \mod m$ and $\gcd(c, m) = 1$. Show that $a \equiv b \mod m$.
   (b) (5 points) Give two examples showing that a is not necessarily equivalent to b above if $\gcd(c, m) \neq 1$.

4. Find all incongruent solutions to each of the following congruences:
   (a) (3 points) $7x \equiv 3 \mod 15$
   (b) (3 points) $6x \equiv 5 \mod 15$
   (c) (3 points) $x^2 \equiv 1 \mod 8$
   (d) (3 points) $x^2 \equiv 2 \mod 7$
   (e) (3 points) $x^2 + x + 1 \equiv 0 \mod 5$

5. (10 points) Find all incongruent solutions to the following congruence: $(10 + x)^{100} - x \equiv 0 \mod 5$

6. (10 points) Let $a \in \mathbb{Z}$. Show that $a^2 - 3$ is not divisible by 4.

7. (10 points) Prove that the following "divisibility tests" work:
   (a) An integer is divisible by 4 if and only if its last two digits are divisible by 4
   (b) An integer is divisible by 9 if and only if the sum of its digits is divisible by 9
   (c) An integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11. (If the digits of n are $n_0 n_1 \dots n_d$ then the alternating sum of its digits is $n_0 - n_1 + n_2 - \cdots$)