Math 4400 Homework 1
Due: Monday, May 22nd, 2017

Feel free to work with your classmates, but everyone must turn in their own assignment. Please make a note of who you worked with on each problem. Also, please give me an estimate of how long this assignment took to complete.

Let me know if you find a typo, or you're stuck on any of the problems.

1. Prove the following statements:
   (a) $\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$, for all integers $n \geq 1$.
   (b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, for all integers $n \geq 1$
   (c) $\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = n + 1$, for all integers $n \geq 1$, where $\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n$ denotes the product

2. The following is an argument that all cows are the same color. We prove this by induction, by setting $P(n) =$"any collection of n cows all have the same color". Clearly, $P(1)$ is true since every cow is the same color as itself. Now let $k \geq 1$ be a natural number and suppose $P(k)$ is true and let S be a set of $k+1$ cows, numbered $1, 2, \dots, k+1$. Then cows 1 through k are all the same color, and cows 2 through $k+1$ are all the same color, by the induction hypothesis. But this means all $k+1$ of our cows are the same color, so we've proven $P(k+1)$. By induction, we've shown $P(n)$ is true or all n, and in particular when n is the number of cows on earth. So we've shown that all cows must be the same color.

   Now, a quick google search shows that there are different colors of cows in the world. What's wrong with the argument above?

3. (a) Prove that any finite, non-empty subset of $\mathbb{Z}$ has a minimum.
   (b) Use part (a) to show that any finite, non-empty subset of $\mathbb{Z}$ has a maximum.
   (c) Use part (b) to show that if $a, b \in \mathbb{Z}$ and $a \neq 0$, then $\gcd(a, b)$ exists and is unique.

4. Compute the following gcd's using the euclidean algorithm:
   (a) $\gcd(1084, 412)$
   (b) $\gcd(1979, 531)$
   (c) $\gcd(305, 185)$

5. Use your work for the above exercise to compute the continued fractions expansions of the following:
   (a) $\frac{1084}{412}$
   (b) $\frac{1979}{531}$
   (c) $\frac{305}{185}$

6. Find the continued fraction expansion of $\sqrt{7}$ and prove it's periodic. (Hint: we learned in class that $\sqrt{7}$ should have a periodic continued fraction. Use a computer or a calculator to guess what it should be, then see if you can prove that's the case by showing $\sqrt{7} - 2$ appears in its own continued fraction expansion, kind of like what we did in class for $\sqrt{2}$)