Injections, Surjections, and Bijections

Math 4400, Summer 2017

Let S and T be two nonempty sets.

Definition. *A function $f : S \to T$ is said to be* one-to-one*, or* injective*, if different inputs get sent to different outputs. More formally, f is injective if it satisfies: $\forall s_1, s_2 \in S$, if $f(s_1) = f(s_2)$, then $s_1 = s_2$. An* "injection" *is an injective function.*

Example. Suppose $S = \{1, 2, 3\}$ and $T = \{a, b, c, d\}$. Then the map $f : S \to T$ defined by $f(1) = a$, $f(2) = b$, and $f(3) = c$ is injective. The map $g : S \to T$ defined by $g(1) = a$, $g(2) = b$, $g(3) = a$ is *not* injective, since $g(1) = g(3)$, even though $1 \neq 3$.

Definition. *A function $f : S \to T$ is said to be* onto*, or* surjective*, if every element of T gets mapped onto. More formally, f is surjective if it satisfies: $\forall t \in T, \exists s \in S$ such that $f(s) = t$. A* "surjection" *is a surjective function.*

Example. Suppose that $S = \{1, 2, 3, 4\}$ and $T = \{a, b, c\}$. Then the map $f : S \to T$ defined by $f(1) = a$, $f(2) = c$, $f(3) = b$, $f(4) = a$ is surjective. The function $g : S \to T$ defined by $g(1) = a$, $g(2) = b$, $g(3) = a$, $g(4) = b$ is *not* surjective, since g doesn't send anything to c.

Definition. *A function $f : S \to T$ is said to be* bijective *if it is both injective and surjective. A* "bijection" *is a bijective function.*

Example. Let $S = \{1, 2, 3\}$ and $T = \{a, b, c\}$. Then the function $f : S \to T$ defined by $f(1) = a$, $f(2) = b$, and $f(3) = c$ is a bijection. Another example is the function $g : S \to T$ defined by $g(1) = c$, $g(2) = b$, $g(3) = a$.

Question. In the above example, how many different bijections are there from S to T?

Fact. If S and T are finite sets, and $\#S > \#T$, then there are no injective functions from S to T. For instance, there are no injective functions from $S = \{1, 2, 3\}$ to $T = \{a, b\}$: an injective function would have to send the three different elements of S to three different elements of T. But T only has two elements. There's just not enough space in T for there to be an injective function from S to T!

By contrapositive, if there exists an injection $f : S \to T$, then $\#S \leq \#T$

Fact. Similarly, if S and T are finite sets, and $\#S < \#T$, then there are no surjective functions from S to T. For instance, if $S = \{1, 2\}$ and $T = \{a, b, c\}$, there's no way to map the two elements of S onto all three elements of T. By contrapositive, if there exists a surjection $f : S \to T$, then $\#S \geq \#T$.

Fact. By combining the above two facts, we see: if S and T are finite sets and there exists a bijection $f : S \to T$, then $\#S = \#T$. (By definition, f is injective, so $\#S \leq \#T$. Also by definition, f is surjective, so $\#S \geq \#T$. Thus $\#S = \#T$)