9/6/22 Agenda: 1.3, linear combos

Start 2.1, linear tranforms.

Warm-up: let $M = \begin{bmatrix} 6 & 2 \\ 1 & -1 \\ 3 & 0 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Find the following, if possible:

a) $M \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ b) $M \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ c) $4 \cdot M$

d) $M \cdot N$ e) $N \cdot M$

Answers

a) $A \cdot B$ part a) is not possible!

size $m \times n$ $n \times l$

$\begin{bmatrix} & \\ & \end{bmatrix}_{3\times 2} \cdot \begin{bmatrix} \\ \end{bmatrix}_{3\times 1}$

b) $\begin{bmatrix} & \\ & \end{bmatrix}_{3\times 2} \cdot \begin{bmatrix} \\ \end{bmatrix}_{2\times 1}$

$\begin{bmatrix} 6 & 2 \\ 1 & -1 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$

c) $\begin{bmatrix} 24 & 8 \\ 4 & -4 \\ 12 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 6 & 2 \\ 1 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} M\begin{bmatrix} 0 \\ 1 \end{bmatrix} & M\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}$

$= \begin{bmatrix} 2 & 8 \\ -1 & 0 \\ 0 & 3 \end{bmatrix}$

e) $\begin{bmatrix} & \end{bmatrix}_{2\times 2} \begin{bmatrix} & \\ & \end{bmatrix}_{3\times 2}$ not possible!

$M \cdot N \neq N \cdot M$

Formula for matrix mult :

if $A \cdot B = C$ then C has size $m \times l$ matrix,

size $m\times n$ $\quad$ $n \times l$

and $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

entry of matrix C in row i and column j

entries in matrix A and matrix B.

for all i and j with $1 \le i \le m$, $1 \le j \le l$.

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§ Matrix form of sys of eq's

Given
$$4x_1 + 2x_2 = 1$$
$$3x_1 + 6x_2 = 4$$
we've been writing this as an augmented matrix

$$\left[\begin{array}{cc|c} 4 & 2 & 1 \\ 3 & 6 & 4 \end{array}\right]$$

Another way:
$$\begin{bmatrix} 4x_1 + 2x_2 \\ 3x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$\parallel$$
$$\begin{bmatrix} 4 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad A\vec{x} = \vec{b}$$

$\vec{x}$ $\quad$ $\vec{b}$

A "coefficient matrix"

Recall :
- if RREF(A) has a pivot in each column, then sys. has either 1 sol'n or 0 solutions.
- if RREF(A) has a column without a pivot, ∞ many sols or 0.

(0 sol'ns $\Longleftrightarrow$ some row in RREF(A) is 0, corresp entry in $\vec{b}$ is not 0)

~~§ Linear Combinations~~

Q: pivot in each column, but no sol's?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= 3 \\ x_2 &= 2 \\ 0x_1 + 0x_2 &= 4 \end{aligned} \quad X$$

Linear Combos ← "combinations"

Recall: if $\vec{v}_1, \dots \vec{v}_n$ of the same dimension, [ ]

a linear combo of $\vec{v}_1, \dots, \vec{v}_n$ is a sum of the form $c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, where $c_1, \dots, c_n \in \mathbb{R}$

e.g. A linear combo of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$:

$$\underbrace{47.2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3.7 \begin{bmatrix} 2 \\ -3 \end{bmatrix}}_{\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 47.2 \\ 3.7 \end{bmatrix}}$$

(linear combos of vectors can be written as a matrix product)

Q a) Which vectors $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ can be written as a linear combo of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

"in" — set of 2-dim'l vectors

b) Which vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combo of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$?

A a) All vectors in $\mathbb{R}^2$!

~~Given~~ Given any $\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

b) When can you write

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$ for some $c_1, c_2 \in \mathbb{R}$?

$$= \begin{bmatrix} c_1 - 3c_2 \\ 2c_1 + c_2 \\ 0 + 0 \end{bmatrix}$$ need $c = 0$!

Subquestion: can every vector $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ be written as a linear combo of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$?

I.e. Does the equation $\begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ always have a solution?

Using row reduction: $\mathrm{RREF}\left(\left[\begin{array}{cc|c} 1 & -3 & a \\ 2 & 1 & b \\ 0 & 0 & 0 \end{array}\right]\right) = ?$

$$= \left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right]$$

there is a solution!

Not saying "0 = 1"

So every vector of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in \mathbb{R}^3$ works.

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§2.1 Notation: $f: A \to B$ "f is a function from the set A to the set B"

So if $x \in A$, $f(x) \in B$

Def A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation (or linear function) if

- $f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$, and
- $f(c\vec{v}) = cf(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$

e.g. Consider $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(\vec{v}) = 2\vec{v}$

This function is linear, because:
- $2(\vec{v}+\vec{w}) = 2\vec{v} + 2\vec{w}$
- $2(c\vec{v}) = c\cdot(2\vec{v})$

Non-example: $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a^2 \\ b \end{bmatrix}$$

This is not linear: $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$

$$\neq$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$