§3.1 "image", "kernel", "span".

If A is an $m \times n$ matrix, then

- $\mathrm{im}(A) = \mathbb{R}^m \iff$ pivot in each row
- $\ker(A) = \{0\} \iff$ pivot in each column

<u>Recall</u>: • $A\vec{x} = \vec{b}$ has <u>no</u> solutions $\iff$ RREF$(A \vdots \vec{b})$ has the row $[0 \cdots 0 | 1]$

• If $A\vec{x} = \vec{b}$ does have a solution:
- unique solution $\iff$ A has a pivot in each col
- infinitely many solutions otherwise

Last time: A is $m \times n$ matrix

$$\mathrm{im}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m \quad \text{"subset"}$$

"image"

$$= \text{span of columns of } A$$

<u>Note</u>: $\vec{v} \in \mathrm{im}(A) \iff \vec{v} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$

$\iff$ the eq. $A\vec{x} = \vec{v}$ has a solution $\vec{x} \in \mathbb{R}^n$

So $\mathrm{im}(A) = \mathbb{R}^m \iff A\vec{x} = \vec{v}$ has a solution for all $\vec{v} \in \mathbb{R}^m \iff \mathrm{rref}(A)$ has a pivot in each row

"multiplying by A is 'surjective'"

<u>e.g.</u> $\mathrm{im}\left(\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right) = ?$

"for which $\vec{v}$ does $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\vec{x} = \vec{v}$ have

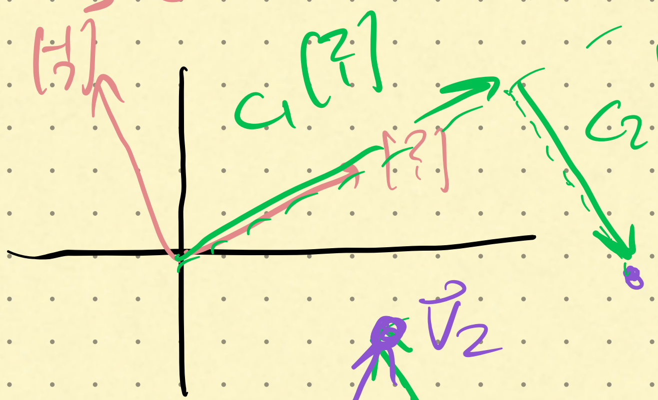
" ... ?"

"span $\left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right) = ?$

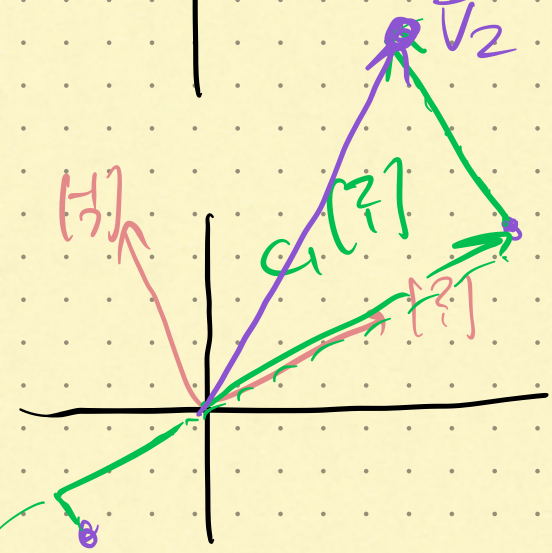
a solution?"

$$\text{rref}\left(\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}\right) = \underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{\text{pivot in each row.}} \rightsquigarrow \text{im}\left(\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}\right) = \mathbb{R}^2$$

$\Rightarrow$ each vector in $\mathbb{R}^2$ is a linear combo of $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}$



$$\vec{v} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\2\end{bmatrix}$$



$\vec{v} = c$

<u>eg</u> $\text{im}\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} = ?$

$\hookrightarrow$ for which $\vec{v}$ does $\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}\vec{x} = \vec{v}$ have a solution?

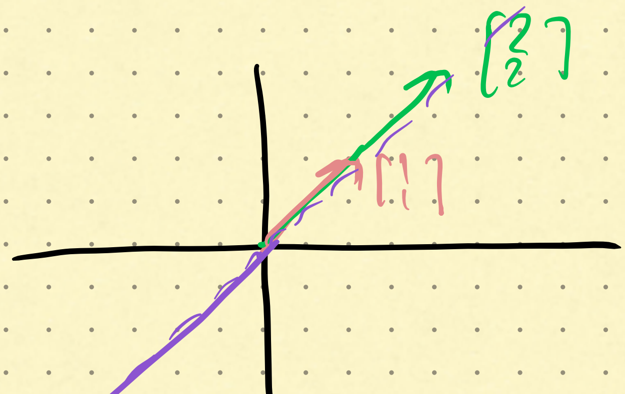


image = the line $y=x$

$$= \left\{ c\begin{bmatrix}1\\1\end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

Algebraic sol'n: $\left[\begin{array}{cc|c} 1 & 2 & y_1 \\ 1 & 2 & y_2 \end{array}\right]$ when is there a sol'n?

RREF: R2-R1: $\left[\begin{array}{cc|c} 1 & 2 & y_1 \\ 0 & 0 & y_2-y_1 \end{array}\right]$ y_2-y_1 better be 0

$\Rightarrow y_2=y_1$ $\vec{v}=\begin{bmatrix} c \\ c \end{bmatrix}$, for some $c\in\mathbb{R}$

<u>kernel</u>: if $T:\mathbb{R}^m \to \mathbb{R}^n$, then

$$\ker(T) = \{x\in\mathbb{R}^m \mid T(\vec{x}) = \vec{0}_n\}$$

vector of all zeros of length n.

if A is a $n\times m$ matrix, then

$$\ker(A) = \{\vec{x}\in\mathbb{R}^m \mid A\vec{x}=\vec{0}_n\}$$

e.g. $A=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ $\ker(A)=?$

↳ solve $A\vec{x}=\vec{0}$

$\text{RREF}(A \mid \vec{0}) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow$ $x_1+x_3=0$
$x_2+x_3=0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \text{ where } x_3 \in \mathbb{R}$$

$$\ker(A) = \left\{ c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

"$\ker(A)$ is spanned by $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$"

(If $\mathrm{RREF}(A) = I_n$, then $\ker(A) = \{\vec{0}\}$)

For each matrix A in Exercises 1 through 13, find vectors that span the kernel of A. Use paper and pencil.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **4.** $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$ **6.** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

30. Give an example of a matrix A such that $\mathrm{im}(A)$ is spanned by the vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y = x/3$ in $\mathbb{R}^2$

#1: Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] \overset{\mathrm{RREF}}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\ker\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

#2 Solve $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$$\rightsquigarrow x_1 + 2\underset{\uparrow}{x_2} + 3\underset{\uparrow}{x_3} = 0 \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

free vars.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x} \in \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)$$