9/20/22

#76. Given $A\begin{bmatrix}1&2\\2&5\end{bmatrix} = \begin{bmatrix}2&1\\1&3\end{bmatrix}$

$$\underbrace{\begin{bmatrix}1&2\\2&5\end{bmatrix}^{-1}\begin{bmatrix}1&2\\2&5\end{bmatrix}}_{I_2}A = \begin{bmatrix}1&3\\2&5\end{bmatrix}^{-1}\begin{bmatrix}2&1\\1&3\end{bmatrix}$$

$$A\underbrace{\begin{bmatrix}1&2\\2&5\end{bmatrix}\begin{bmatrix}1&2\\2&5\end{bmatrix}^{-1}}_{\begin{bmatrix}1&0\\0&1\end{bmatrix}} = \begin{bmatrix}2&1\\1&3\end{bmatrix}\begin{bmatrix}1&2\\2&5\end{bmatrix}^{-1}$$

$$A = \begin{bmatrix}2&1\\1&3\end{bmatrix}\begin{bmatrix}1&2\\2&5\end{bmatrix}^{-1} = \begin{bmatrix}8&-3\\-1&1\end{bmatrix}$$

Caution: if we mult. on the left: by A^{-1}:

$$\underbrace{\begin{bmatrix}1&2\\2&5\end{bmatrix}^{-1}A\begin{bmatrix}1&2\\2&5\end{bmatrix}}_{??} = \begin{bmatrix}1&2\\2&5\end{bmatrix}^{-1}\begin{bmatrix}2&1\\1&3\end{bmatrix}$$

67: $(A+B)^2 \overset{?}{=} A^2 + 2AB + B^2$ (A and B are invertible $n\times n$ matrices)

No: $(A+B)(A+B) = (A+B)A + (A+B)B$

$= A^2 + BA + AB + B^2$ may not equal $A^2 + 2AB + B^2$!

70 $(A^2)^{-1} \overset{?}{=} (A^{-1})^2$ Yes!

$$(A^{-1})^2 \cdot A^2 = A^{-1}\underbrace{A^{-1}\cdot A}_{I_n}\cdot A = \underbrace{A^{-1}A}_{I_n} = I_n$$

$(A^2)^{-1}$

$(A^{-1})^2$ is the inverse of A^2

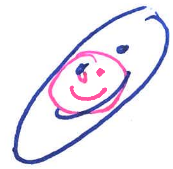
$(A^{-1})^2 = (A^2)^{-1}$ ✓

Associativity: $(AB)C = A(BC) \rightsquigarrow (AB)(CD) = A(BC)D$

#72 : $ABA^{-1} \stackrel{?}{=} B$ No!

e.g. $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}^{-1} = \begin{bmatrix}1&0\\0&2\end{bmatrix} \neq \begin{bmatrix}2&0\\0&1\end{bmatrix}$

Recall: Stretching diagonally?



$$\left(\text{Rot}_{\pi/4}\right)\left(\begin{matrix}\text{stretch}\\\text{horiz}\end{matrix}\right)\left(\text{Rot}_{-\pi/4}\right) \neq \left(\begin{matrix}\text{stretch}\\\text{horiz.}\end{matrix}\right)$$

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§2.4. Determinants.

For a 2x2 matrix $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, A is <u>not</u> invertible

$\iff$ RREF(A) has a row of zeros

$\iff$ $[a \;\; b]$ is a constant multiple of $[c \;\; d]$ e.g. $\begin{bmatrix}2&6\\4&12\end{bmatrix} \to \begin{bmatrix}2&6\\0&0\end{bmatrix}$

$\iff \dfrac{a}{b} = \dfrac{c}{d}$ $(b, d \neq 0)$ eg: $\dfrac{2}{6} = \dfrac{4}{12}$

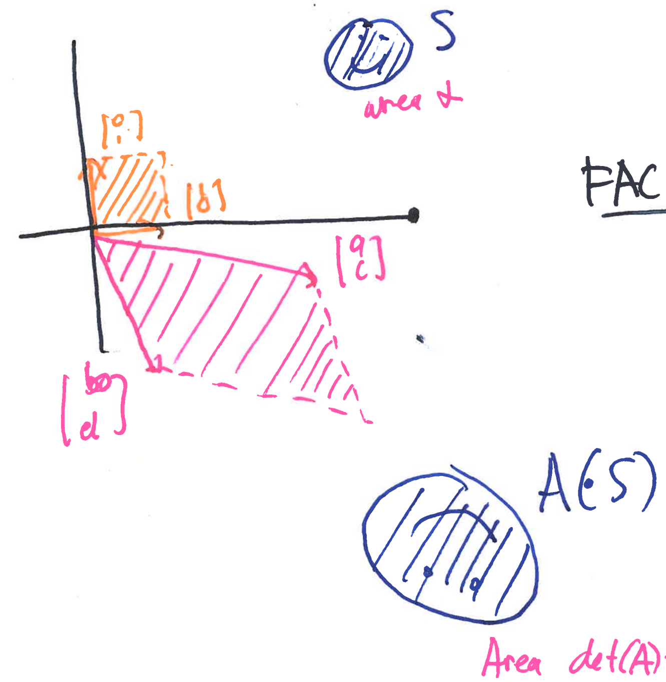
$\iff ad = cb \iff ad - cb = 0$

Thm $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ is invertible $\iff \underbrace{ad - cb}_{\text{"determinant of } A\text{"}} \neq 0.$

det(A)

Geometrically

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$



A sends the orange square to the pink parallelogram.

FACT $\det(A)$ = area of pink parallelogram.

Fact A sends any shape of area α to a shape of area $\det(A)\cdot\alpha$

FACT this generalizes to higher dimensions.

e.g. if A is a 3×3 matrix, then $\det(A)$ is the volume of the shape that A sends a $1\times1\times1$ cube to.

etc. for 4-dims, 5-dims, . . .

In higher dims, $\det(A)$ is a pain to compute.

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

(Ch. 6)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh + bfg - bdi + \cdots$$

"sign of the permutation"

Ch. 3:

§3.1: Goal: • For any $m \times n$ matrix A,

A has a pivot in each col $\Leftrightarrow \ker(A) = \{0\}$

• A has a pivot in each row $\Leftrightarrow \operatorname{im}(A) = \mathbb{R}^n$

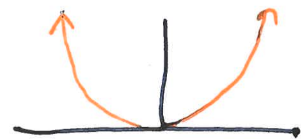
(summary p.118)

Def Given a function $f: X \to Y$,

image of f = $\operatorname{im}(f) = \{f(x) \mid x \in X\}$

"range of f"

e.g. $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$



$$\operatorname{im}(f) = \{f(x) \mid x \in \mathbb{R}\} = \{x^2 \mid x \in \mathbb{R}\} = [0, \infty)$$

e.g. image of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$? (ie. image of $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \vec{x}$)

$$\operatorname{im}\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \right\}$$

$$= \left\{ \underbrace{x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{linear combo}} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}$$

$$= \left\{ \text{linear combos of } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

In general, the image of any matrix is the set of linear combos of its columns. → "span" of its columns.

__Def__ if $\vec{v}_1, \ldots, \vec{v}_p \in \mathbb{R}^n$, the span of $\vec{v}_1, \ldots, \vec{v}_r$ is the set of linear combos of $\vec{v}_1, \ldots, \vec{v}_r$.

So $\text{im}(A) = \text{span}(\text{columns of } A)$