Agenda
- another projection formula
- Start 6.1: determinants

Last time: ∘ $(\text{im } A)^\perp = \ker(A^T)$

∘ Also: To approximately solve "$A\vec{x} \approx \vec{b}$", instead solve

$$A\vec{x} = \text{proj}_{\text{im}A}\vec{b} \iff A^TA\vec{x} = A^T\vec{b}$$

Let's go further: suppose the columns of A are lin indep.

Then thm 5.4.2: A^TA is invertible.

So $A\vec{x} = \text{proj}_{\text{im}A}\vec{b} \iff \vec{x} = (A^TA)^{-1}A^T\vec{b}$

Apply A to both sides:

$$\underbrace{A\vec{x}}_{\text{proj}_{\text{im}A}\vec{b}} = \boxed{A(A^TA)^{-1}A^T}\vec{b}$$

New projection formula: let $\vec{v}_1, \ldots, \vec{v}_d$ be a basis for $V \subseteq \mathbb{R}^n$.

Let $A = [\vec{v}_1 \cdots \vec{v}_d]$. Then the matrix for proj_V is $A(A^TA)^{-1}A^T$

Compare: if $\vec{v}_i$ are an ONB for V, we saw AA^T is the matrix for proj_V

§ 6.1: Determinants

2×2 case: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin. transform with matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

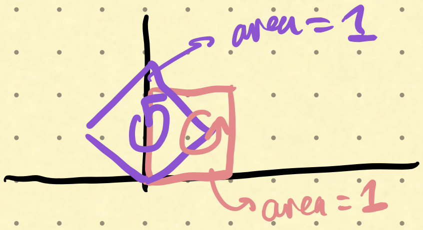
Def $\det(T) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

Geometrically: $|\det(T)|$ = the amount that T scales area

= area of T(1×1 square)

$\det T > 0 \iff T$ "preserves orientation"

eg $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $45°$ CCW

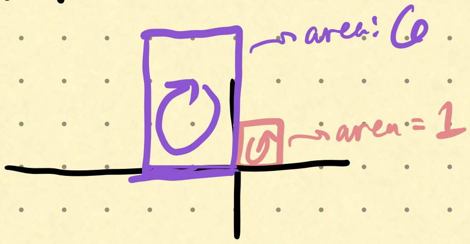


- T preserves orientation: arrow still pts CCW after applying T
- T multiplies areas by 1.

$\Rightarrow \det(T) = +1$

Algebraically: $\det(T) = \det\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \left(\frac{-1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$

eg $T: \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$



T reverses orientation: $\det T < 0$

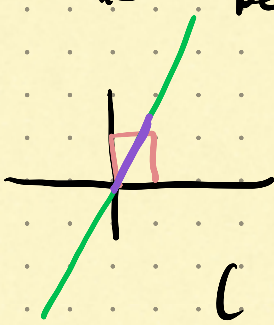
T multiplies areas by 6

$\Rightarrow \det(T) = -6$

Algebraically:
$$\det(T) = \det\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} = -2 \cdot 3 - 0 \cdot 0 = -6$$

exc let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto line spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $\det(T)$.

Geometrically:



$T(\square)$ = line segment on green line.

$\Rightarrow \det T = 0$

(T invertible $\Leftrightarrow$ $\det \neq 0$)

algebraically:
$$\det T = \det\left(\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$$
$$= \frac{1 \cdot 4}{5 \cdot 5} - \frac{2 \cdot 2}{5 \cdot 5} = 0$$

$n \times n$ case:

We can define the determinant of any (square) $n \times n$ matrix, or equivalently of any transform $T: \mathbb{R}^n \to \mathbb{R}^n$

Facts:

- T multiplies "volumes" by $|\det(T)|$
- M invertible $\iff \det(M) \neq 0$
- Lots of formulas for computing.

Recursive formula for det: (Thm 6.2.10)

Let M be an $n \times n$ matrix. For all i, j, $1 \leq i, j \leq n$,
let $m_{ij} = (i,j)$ entry of M, M_{ij} = matrix M with ith row, jth column deleted

(eg. $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. $m_{21} = 4$ $M_{21} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$)

if M is $n \times n$ matrix, then M_{ij} is an $(n-1) \times (n-1)$ matrix

<u>Formula</u>: $\det M = \sum_{j=1}^{n} (-1)^{i+j} m_{ij} \cdot \det(M_{ij})$

$= \sum_{i=1}^{n} (-1)^{i+j} m_{ij} \det(M_{ij})$

eg find $\det \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$. Let's expand along top row:

$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$

$\det = (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix} + (-1)^{1+2} \cdot 2 \cdot \det \begin{bmatrix} 4 & 5 \\ 0 & 3 \end{bmatrix} + (-1)^{1+3} \cdot \underline{0} \cdot \det \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$= 1 \cdot 1 \cdot (0 \cdot 3 - 5) + (-1) \cdot 2 \cdot (4 \cdot 3 - 5 \cdot 0) + 0$

$= -5 - 24 = -29$

<u>exc</u> find det of that same matrix $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$ expanding along a different row/column.

eg. 2^{nd} col:

$$(-1)^{1+2} \cdot 2 \det \begin{bmatrix} 4 & 5 \\ 0 & 3 \end{bmatrix} + (-1)^{2+2} \cdot 0 \cdot \det[\ ] + (-1)^{3+2} \cdot 1 \cdot \det \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$$

$$= -1 \cdot 2 \cdot (4 \cdot 3 - 0 \cdot 5) + 0 + -1 \cdot 1 \cdot (1 \cdot 5 - 4 \cdot 0) = -29$$