Math 214 5.3 worksheet

1. If A is any matrix, recall that A^T denotes the *transpose* of A. This is defined as the matrix where the columns of A^T are the rows of A, and vice-versa.

   (a) Write down the transposes of the following matrices:

   $$A = \begin{bmatrix} \pi & e \\ \sqrt{2} & 1/3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 7 \end{bmatrix}$$

   (b) What is $(A^T)^T$ (the transpose of A-transpose)? In general, if E is any matrix, what is $(E^T)^T$?

   (c) Suppose E is a matrix with m rows and n columns. How many rows and columns does E^T have?

2. Let $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$. Compute the dot product $\vec{v} \cdot \vec{w}$ and the matrix product $\vec{v}^T \vec{w}$. What do you notice?

3. Let $A = \begin{bmatrix} 0 & 4 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 2 & 0 \end{bmatrix}$

   (a) Find $A^T B$.

   (b) Let $\vec{v}_1, \vec{v}_2$ be the columns of A and let $\vec{w}_1, \vec{w}_2$ be the columns of B. Compute all the dot products $\vec{v}_i \cdot \vec{w}_j$. What do you notice?

   (c) Let $A = [\vec{a}_1 \dots \vec{a}_r]$ and $B = [\vec{b}_1, \dots, \vec{b}_s]$, where $\vec{a}_i$ and $\vec{b}_j$ are all vectors in $\mathbb{R}^n$. Is the matrix product $A^T B$ well-defined? (That is, do these two matrices have the correct numbers of rows/columns so that this product is defined?) If so, fill in the blanks: the i, j entry of $A^T B$ (meaning the entry in the ith row and jth column) is the dot product of ______ and ______

4. (a) Let $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Compute $A^T A$.

(b) Suppose B is an $m \times n$-matrix with orthonormal columns. What can you say about $B^T B$? (Hint: the matrix A from part (a) has orthonormal columns!)

(c) Suppose $\vec{u}_1, \ldots, \vec{u}_n$ is some orthonormal basis of $\mathbb{R}^n$. Let $Q = [\vec{u}_1 \ \cdots \ \vec{u}_n]$. In other words the columns of Q are the vectors $\vec{u}_1, \ldots, \vec{u}_n$. Is Q invertible? If so, is there a quick way to find Q^{-1}?

5. Let

$$\vec{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix},$$

and let $V \subseteq \mathbb{R}^4$ be the subspace spanned by $\vec{u}_1$ and $\vec{u}_2$.

(a) Let $\vec{w} = \begin{bmatrix} 2 & 3 & 4 & 0 \end{bmatrix}^T$. Compute $\text{proj}_V(\vec{w})$,

(b) Let $Q = [\vec{u}_1 \ \vec{u}_2]$ be the matrix whose columns are $\vec{u}_1$ and $\vec{u}_2$. Compute $QQ^T\vec{w}$. What do you notice?