5.2: Graham-Schmidt algo

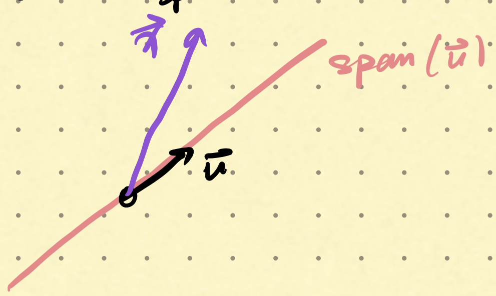
Recall: a set $\{\vec{u}_1, \dots, \vec{u}_d\}$ is orthonormal if

$$u_i \cdot u_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

If $V = \text{span}\{\vec{u}_1, \dots, \vec{u}_d\}$, then

$$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_d \cdot \vec{x})\vec{u}_d$$

eg. if $\vec{u}$ is a unit vector, then $\{\vec{u}\}$ is an ONB for $\text{span}(\vec{u})$. Then $\text{proj}_{\text{span}(\vec{u})}(\vec{x}) = (\vec{u} \cdot \vec{x})\,\vec{u}$



Q how do we find ONBs?

A Graham-Schmidt algo

input: a basis $\{\vec{v}_1, \dots, \vec{v}_d\}$ of a subspace $V \leq \mathbb{R}^n$

output: an ONB $\{\vec{u}_1, \dots, \vec{u}_d\}$ of V

Idea:
- first find an ONB $\{\vec{u}_1\}$ for $\text{span}\{\vec{v}_1\}$,
- then find an ONB $\{\vec{u}_1, \vec{u}_2\}$ for $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

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- find an ONB $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d\}$ for $\text{span}\{\vec{v}_1, \dots, \vec{v}_d\}$.

We get $\vec{u}_i$ from $\vec{u}_1, \dots, \vec{u}_{i-1}$ using orthogonal projection

E.g. let $V = \text{span}\left(\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}} \right) \leq \mathbb{R}^4$

Find an ONB for V

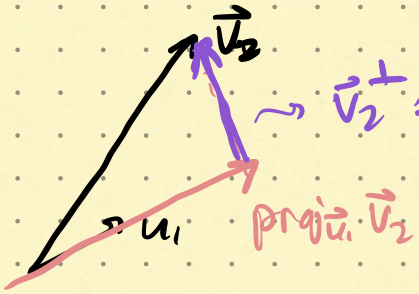
First, find an ONB for span$(\vec v_1)$: $\frac{1}{\|\vec v_1\|}\vec v_1 = \vec u_1$

$\Rightarrow \vec u_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$

<u>Next</u>: use this to find an ONB for span$(\vec v_1, \vec v_2)$:

$\vec u_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec u_2 = ?$ $\vec v_2$ is not orthogonal to $\vec u_1$

Replace $\vec v_2$ with something ortho to $\vec u_1$



$\rightsquigarrow \vec v_2^{\perp} = \vec v_2 - \text{proj}_{\vec u_1}(\vec v_2)$

Because u_1 is a unit vector,

$\text{proj}_{u_1}(\vec v_2) = (\vec u_1 \cdot \vec v_2)\vec u_1$

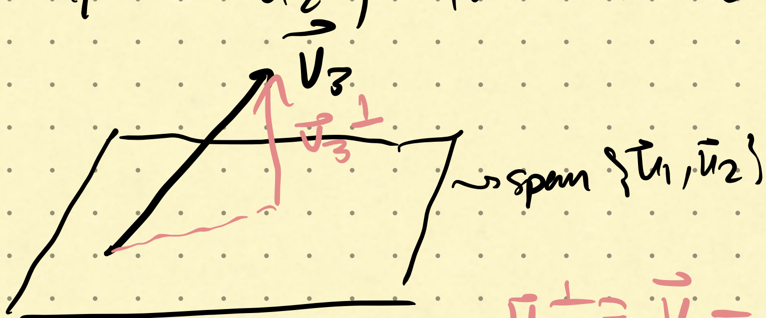
$\Rightarrow \vec v_2^{\perp} = \vec v_2 - (\vec u_1 \cdot \vec v_2)\vec u_1 = \vec v_2 - 10\vec u_1 = \begin{bmatrix}-4\\4\\4\\-4\end{bmatrix}$

Now: $\vec u_1, \vec v_2^{\perp}$ are ortho. and span the span$(\vec v_1, \vec v_2)$

But: $\vec v_2^{\perp}$ is not a unit vector.

$\Rightarrow \vec u_2 = \frac{1}{\|\vec v_2^{\perp}\|}\vec v_2^{\perp} = \frac{1}{8}\begin{bmatrix}-4\\4\\4\\-4\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$

To find $\vec u_3$: Replace v_3 w/ something ortho to $\vec u_1$ and $\vec u_2$, then divide that vector by its length.



$\vec v_3^{\perp} = \vec v_3 - \underbrace{\text{proj}_{\{u_1,u_2\}}\vec v_3} = \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$

$\boxed{\text{proj}_{\{u_1,u_2\}}(\vec v) = (\vec u_1 \cdot \vec v)\vec u_1 + (\vec u_2 \cdot \vec v)\vec u_2}$ $\cancel{= \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}} \cancel{\neq} \cancel{v_3^{\perp}}$

Now: $\vec{u}_1, \vec{u}_2, v_3^\perp$ are all ortho to each other,

So we replace replace $\vec{v}_3^\perp$ with $\frac{1}{\|\vec{v}_3^\perp\|}\vec{v}_3 = \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} = \vec{u}_3$

Answer: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}\right\},$

EXC: find ONB for span $\underbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}2\\2\\1\\-1\end{bmatrix}}_{\vec{v}_3}$

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$\vec{u}_2 = ?$ $\quad \vec{v}_2^\perp = \vec{v}_2 - \underbrace{(\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1}_{proj_{u_1}(\vec{v}_2)}$, $\quad \vec{u}_2 = \frac{1}{\|\vec{v}_2^\perp\|}\vec{v}_2^\perp = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

$\vec{u}_3 = ?$ $\quad \vec{v}_3^\perp = \vec{v}_3 - \underbrace{\left((\vec{v}_3 \cdot \vec{u}_1)\,\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\,\vec{u}_2\right)}_{proj_{\{u_1, u_2\}}\vec{v}_3}$, $\quad \vec{u}_3 = \frac{1}{\|\vec{v}_3^\perp\|}\vec{v}_3$

$= \frac{1}{2\sqrt{5}}\begin{bmatrix}3\\1\\-1\\-3\end{bmatrix}$

QR factorization

$$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix}1 & 1 & 2\\1 & 0 & 2\\1 & 0 & 1\\1 & 1 & -1\end{bmatrix}$$

$$Q = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3] = \frac{1}{2}\begin{bmatrix}1 & 1 & 3/\sqrt{5}\\1 & -1 & 1/\sqrt{5}\\1 & -1 & -1/\sqrt{5}\\1 & 1 & -3/\sqrt{5}\end{bmatrix}$$

Recall: for any $\vec{x}$, $Q[\vec{x}]_Q = \vec{x}$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

$$\Rightarrow A = Q\begin{bmatrix} [v_1]_Q & [v_2]_Q & [v_3]_Q \end{bmatrix}$$

$$[v_1]_Q = \begin{bmatrix} v_1 \cdot u_1 \\ v_1 \cdot u_2 \\ v_1 \cdot u_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$[v_2]_Q = \begin{bmatrix} v_2 \cdot u_1 \\ v_2 \cdot u_2 \\ v_2 \cdot u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[v_3]_Q = \begin{bmatrix} v_3 \cdot u_1 \\ v_3 \cdot u_2 \\ v_3 \cdot u_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ \sqrt{5} \end{bmatrix}$$

$$\Rightarrow A = Q\underbrace{\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}}$$

something upper-triangular

any matrix w/ lin indep columns

orthonormal columns

Def of magnitude: if $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ then $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$