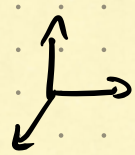
§3.1 : orthonormal sets

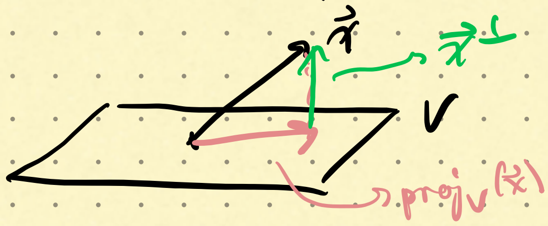
**Def** a set of vectors $\{\vec{u}_1, \ldots, \vec{u}_d\}$ is called orthonormal if $\|\vec{u}_i\| = 1$ for all $i$, and $\vec{u}_i$ is perp. to $\vec{u}_j$ whenever $i \neq j$



Alternatively, $$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Last time: if $\mathcal{O} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis (ONB) for $\mathbb{R}^n$, $$[\vec{x}]_{\mathcal{O}} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \\ \vdots \\ \vec{u}_n \cdot \vec{x} \end{bmatrix}$$

**Orthogonal Projections:** Suppose $V \subseteq \mathbb{R}^n$ is a subspace. Let $\{\vec{u}_1, \ldots, \vec{u}_d\}$ be an ONB of $V$. We can use this to compute $\operatorname{proj}_V(\vec{x})$




**Def** $\operatorname{proj}_V(\vec{x})$ is the (unique!) vector in $V$ such that $\underbrace{\vec{x} - \operatorname{proj}_V(\vec{x})}_{\vec{x}^\perp}$ is orthogonal to $V$

**Formula:** $$\operatorname{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + \cdots + (\vec{u}_d \cdot \vec{x})\vec{u}_d$$

Why? We know $\vec{x} = \underbrace{c_1\vec{u}_1 + \cdots + c_d\vec{u}_d}_{\text{something in } V} + \underbrace{\vec{x}^\perp}_{\text{something ortho. to } V}$ for some $c_1, \ldots, c_d \in \mathbb{R}$

$$\vec{x} \cdot \vec{u}_i = c_1 \underbrace{\cancel{\vec{u}_1 \cdot \vec{u}_i}}_{0} + \cdots + c_i \underbrace{\cancel{\vec{u}_i \cdot \vec{u}_i}}_{1} + \cdots + c_d \underbrace{\cancel{\vec{u}_d \cdot \vec{u}_i}}_{0} + \underbrace{\cancel{\vec{x}^\perp \cdot \vec{u}_i}}_{0} \quad \text{for all } i$$

$\vec{x}^\perp$ is ortho. to everything in $V$

$= c_i$

erc

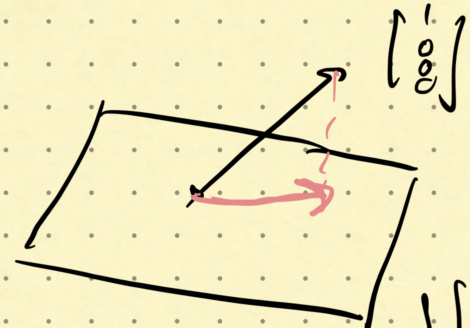
#28

Find the orthogonal projection of

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

onto the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \quad \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}.$$




$V = \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right\}$

not length 1.

length of $\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \sqrt{\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix}}$

$= \sqrt{4} = 2$

Make them length 1 by dividing by 2.

They are already orthogonal: $\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} = 0$

etc.

$\Rightarrow \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$ is an ONB of $V$

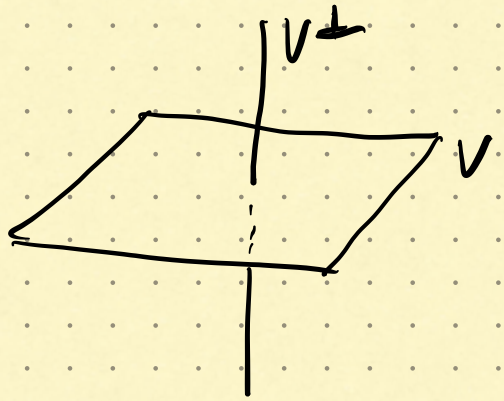
$\Rightarrow \text{proj}_V\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \left(\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \dots$ etc

$= \begin{bmatrix}3/4\\1/4\\-1/4\\1/4\end{bmatrix}$

$\left(\frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\right) \cdot \left(\frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\right) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \underline{\underline{1}}$

# Ortho complement

Let $V \subseteq \mathbb{R}^n$ be a subspace. The ortho complement of $V$ is $V^\perp = \{\vec{w} \in \mathbb{R}^n \mid \vec{w} \cdot \vec{v} = 0 \ \forall \vec{v} \in V\}$

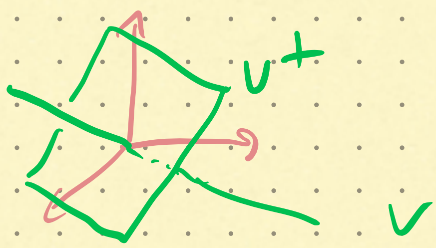



**exc** let $V = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^3$

a) Find a matrix $A$ whose kernel is $V^\perp$

b) Find a basis for $V^\perp$

a)




$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V^\perp \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\iff 2x + z = 0$$

$$\iff \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker\left(\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}\right)$$

b) $V^\perp = \ker\left(\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}\right)$ so we want to find a basis for $\ker\left(\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}\right) \rightsquigarrow$ $2x + z = 0$

$x = -\frac{1}{2} z$
$y = y$
$z = z$

$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$

a basis for $V^\perp$

**Facts about $V^\perp$:** if $V \subseteq \mathbb{R}^n$,

- $\dim V + \dim V^\perp = n$
- $V \cap V^\perp = \{0\}$
- $(V^\perp)^\perp = V$