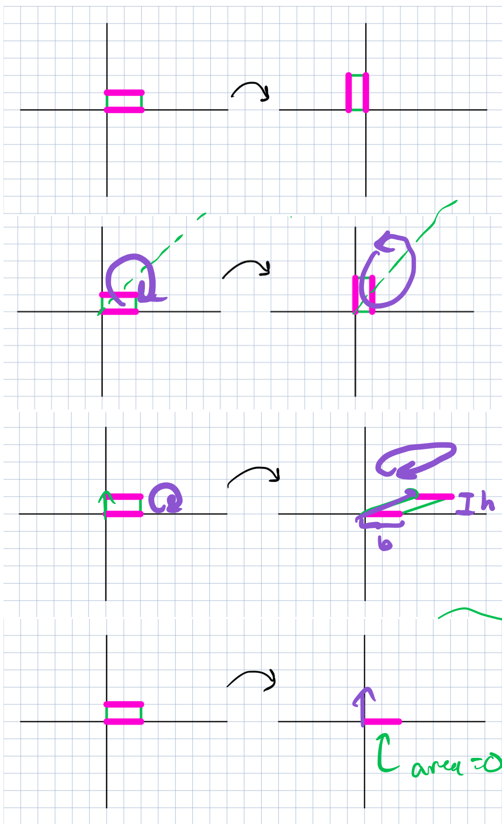
$\begin{bmatrix}1\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}-1\\0\end{bmatrix}$

Midterm 2 review

November 2, 2022

1. Find the determinants of the linear transformations depicted below:



matrix

$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

rotation by 90° ccw

⇒ area is preserved, orientation preserved

⇒ det = +1

$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

reflect across $y=x$

⇒ area is preserved orientation is reversed

⇒ det = −1

Shear

$\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}$

area = 2·1 = 2

- area is preserved
- orientation is preserved

⇒ det = +1

$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

det = 0

. projection onto x-axis.

- not invertible

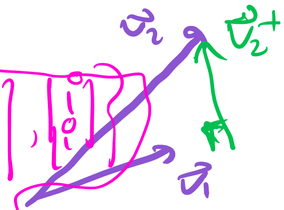
⇒ det = 0

2. Let $M = \begin{bmatrix} 1 & 6 \\ 1 & 4 \\ 1 & 6 \\ 1 & 4 \end{bmatrix}$. Find an orthonormal basis for the image of M.

lin indep

$\text{im}(M) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 6 \\ 4 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 6 \\ 4 \\ 6 \\ 4 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$



Gram-Schmidt:

$$\vec{u}_1 = \frac{1}{\sqrt{1^2+1^2+1^2+1^2}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 6 \\ 4 \\ 6 \\ 4 \end{bmatrix} - \frac{1}{2}(6+4+6+4) \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

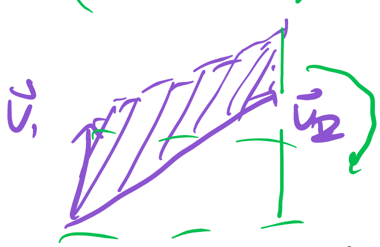
$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

3. Let V be the image of the matrix M above. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Find $\text{proj}_V(\vec{v})$.

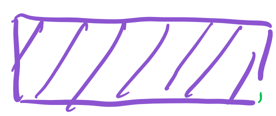
$$(\vec{u}_1 \circ v)\vec{u}_1 + (\vec{u}_2 \circ v)\vec{u}_2$$

$$= \left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

$\det[\vec{v}_1 \ \vec{v}_2] =$



$\det[\vec{v}_1 \ \vec{v}_2^{\perp}] =$

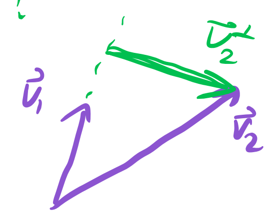


same area!

9. If $\vec{v}_1$ and $\vec{v}_2$ are linearly independent vectors in $\mathbb{R}^2$, what is the relationship between $\det\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ and $\det\begin{bmatrix} \vec{v}_1 & \vec{v}_2^{\perp} \end{bmatrix}$, where $\vec{v}_2^{\perp}$ is the component of $\vec{v}_2$ orthogonal to $\vec{v}_1$?

$\det[\vec{v}_1 \ \vec{v}_2]$

vs



$\det[\vec{v}_1 \ \vec{v}_2]$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \text{proj}_{v_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_1 \circ \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

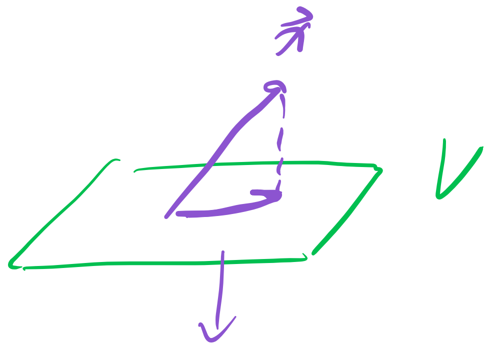
$$[\vec{v}_1 \ \ \vec{v}_2] \xrightarrow{\text{subtract } \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} \text{ times col 1 from col 2}} [\vec{v}_1 \ \ \vec{v}_2^{\perp}]$$

det. does not change!

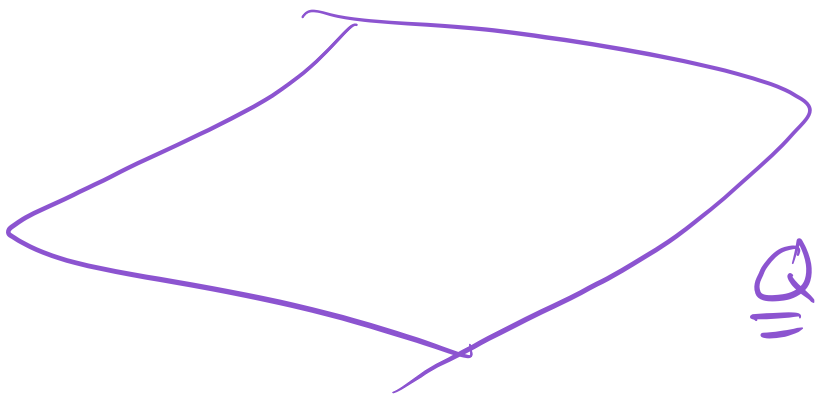
- $\ker(A^T) = \operatorname{im}(A)^\perp$

$\operatorname{im}(A^T) = \ker(A)^\perp$??

- no least squares on midterm!
- $\mathcal{B}$-coords are not a focus
- $\vec{v}^{\parallel}$ projection of V



$\operatorname{proj}_V(\vec{x}) = \vec{x}^{\parallel}$



$V = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$

$\underline{\underline{Q}}$ basis for $V^\perp$?

basis for $\operatorname{im}[\vec{v}_1, \dots, \vec{v}_n]^\perp$

$= \ker \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$