§8.3 Singular value decomposition

A square matrix A is symmetric $\iff A = A^T$

$\iff A$ has an ON eigenbasis $\iff A = SDS^T$

orthogonal diagonal

A symmetric $\iff A = SDS^T$

$\Leftarrow$ short calc.

$\Rightarrow$ spectral theorem.

Why "spectral"? <u>Def</u> the <u>spectrum</u> of a matrix is the set of its eigenvalues.

What about other matrices?

<u>Thm</u> every matrix (with real entries) has a <u>singular value decomposition</u>: $M = U \Sigma V^T$

diagonal (scalar)

orthogonal matrices

(Σ has the same shape as M, so may not be square.
But if $\Sigma = (\sigma_{ij})$, then $\sigma_{ij} = 0$ when $i \neq j$)

<u>Remark</u>: there's a version which also works for matrices w/ complex entries

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singular value decomposition of {{1,2},{3,4},{5,6}}

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD | EXAMPLES | UPLOAD | RANDOM

Input

singular value decomposition $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

Result

Exact forms

$M = U.\Sigma.V^\dagger$

where

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$U = \begin{pmatrix} 0.229848 & 0.883461 & 0.408248 \\ 0.524745 & 0.240782 & -0.816497 \\ 0.819642 & -0.401896 & 0.408248 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 9.52552 & 0 \\ 0 & 0.514301 \\ 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.619629 & -0.784894 \\ 0.784894 & 0.619629 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 0.22 & \cdots \\ 0.52 & \\ 0.81 & \end{pmatrix} \begin{pmatrix} 9.5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.6 & \cdots \\ -0.7 & \cdots \end{pmatrix}$$

right sing. vecs

"left singular vectors"

"singular values"

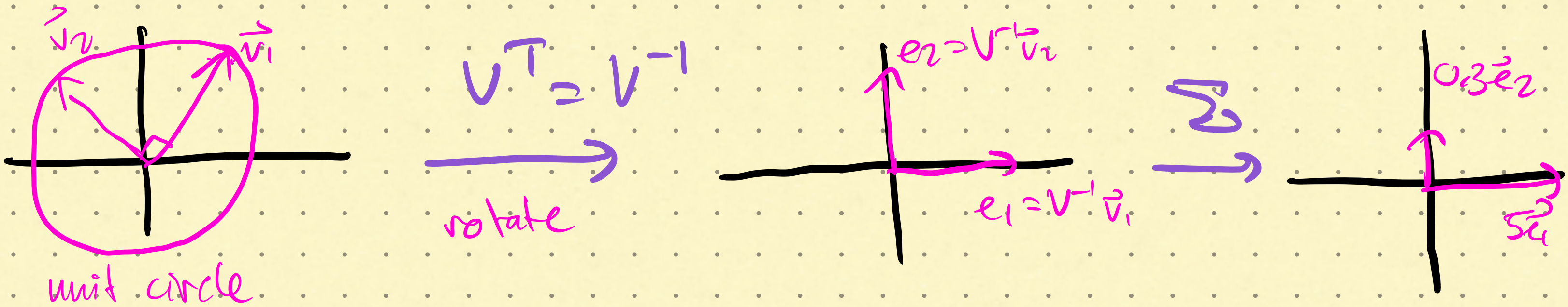
can always arrange so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$

Why do we care?

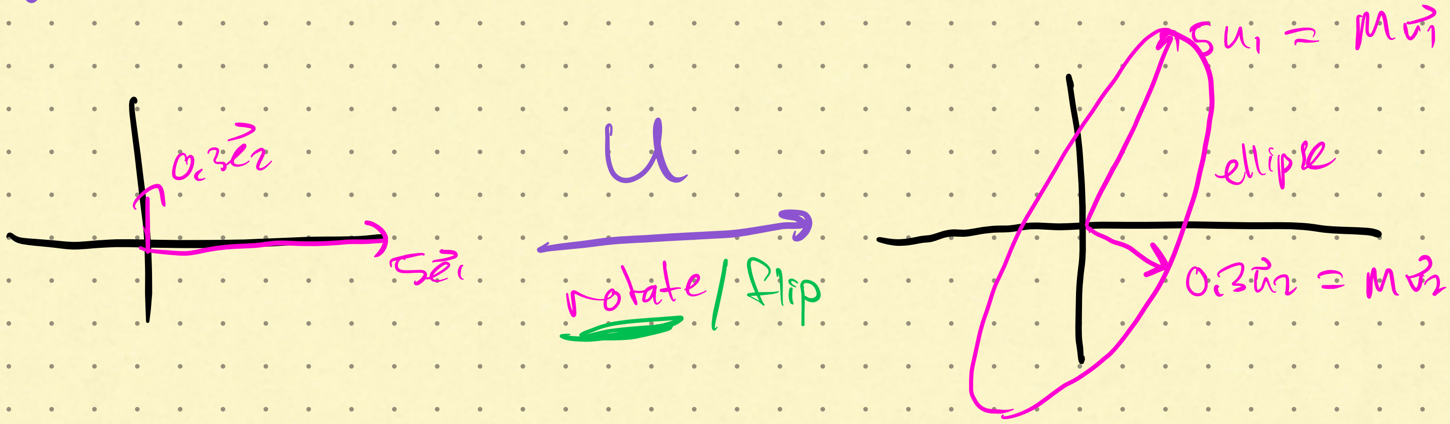
Geometrically:

$$\text{e.g. } \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \approx \begin{pmatrix} 0.4 & 0.9 \\ 0.9 & -0.4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 0.3 \end{pmatrix} \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{pmatrix} \begin{matrix} \leftarrow v_1^T \\ \leftarrow v_2^T \end{matrix}$$

$$U = [\vec{u}_1 \ \vec{u}_2] \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \qquad V^T, \quad V = [\vec{v}_1 \ \vec{v}_2]$$



$$V^{-1}V = I_2 \rightsquigarrow V^{-1}[\vec{v}_1 \ \vec{v}_2] = [\vec{e}_1 \ \vec{e}_2]$$



So: any linear transform is a combination of rotations/flips and scalings.

- left singular vectors give us axes of the ellipse where M sends the unit circle.

How do we compute SVD?

let $A = U \Sigma V^T$ (Recall: U, V orthogonal, Σ "almost" diagonal)

U, Σ, V^T unknown

Then $\underline{A^TA} = (U\Sigma V^T)^T U\Sigma V^T$

symmetric

$= (V^T)^T \underline{\Sigma^T} \underline{U^T \cdot U}\Sigma V^T$

$U^TU = I$

$= V \underbrace{\Sigma^T \Sigma}_{\text{diagonal matrix!}} V^T$ diagonalization.

$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$

$\Sigma^T = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}$

eg. $\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$

This is a diagonalization of A^TA!

$\Rightarrow$ the columns of V are eigenvectors of A^TA!

The eigenvalues of A^TA are the numbers σ_i^2, where σ_i are the nonzero entries of Σ (singular values of A)

What about U?

- one approach: a similar calculation shows that cols of U are eigenvects of AA^T (caution)
- another approach: $\underline{A\vec{v}_i = \sigma_i u_i}$. Solve for $\vec{u}_i$ using the σ_i and $\vec{v}_i$ we already found.