Math 214 7.3 worksheet 2

$\geq$ $\leq$

We saw that the algebraic multiplicity of an eigenvalue is always $\underline{\ \geq\ }$ than its geometric multiplicity.

1. For each of the following matrices find: the characteristic polynomial, the algebraic multiplicity of 2, and the geometric multiplicity of 2:

(a) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

char poly: $\det(A - xI_5) = \det\begin{pmatrix} 2-x & 1 & 0 & 0 & 0 \\ 0 & 2-x & 1 & 0 & 0 \\ 0 & 0 & 2-x & 1 & 0 \\ 0 & 0 & 0 & 2-x & 1 \\ 0 & 0 & 0 & 0 & 2-x \end{pmatrix}$

$= (2-x)^{5}$ alg mult$(2) = 5$

Geom mult$(2) = \dim(E_2) = \dim(\ker(A - 2I_5)) = \dim \underbrace{\ker\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{span}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}} = 1$

$\lambda = 2$ has only one lin. indep. eigenvector.

(b) $B = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ (the (2, 3) entry is the only one that changed)

char poly: still $(2-x)^5$

$E_2 = \ker(B - 2I_5) = \ker\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right\}$

two lin. indep. eigenvectors for $\lambda = 2$.
$\rightarrow$ g. mult$(2) = 2$

(c) $B = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

char poly: still $(2-x)^5$

$E_2 = \ker(B - 2I_5) = \ker\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathbb{R}^5$

$\rightsquigarrow$ g. mult$(2) = \dim(\mathbb{R}^5) = 5$

2. Suppose D is a matrix with characteristic polynomial $p_D(x) = (x-2)^4(x+1)^2(x^2+1)$. What can you say about the geometric multiplicity of $\lambda = 2$ for D? $1 \leq$ g mult$(2) \leq$ a mult$(2) = 4$

While we're at it, what are the dimensions of the matrix D? What is the determinant of D?

dimensions of D: $\deg p_D(x) = 8 \rightsquigarrow D$ is an 8×8 matrix

$\det(D) = p_D(0) = (-2)^4 \cdot (1)^2 \cdot (0^2+1) = 16$

Let's prove this fact!

Theorem If A is a square matrix and λ is an eigenvalue of A, then $\text{gmult}(\lambda) \leqslant \text{algmult}(\lambda)$

Proof idea: (1) we can "partially diagonalize" A using the eigenvectors in E_λ, (2) the theorem is clear for this partially diagonal matrix, and (3) it follows that the theorem holds for A.

Proof

- Suppose λ is an eigenvalue of A with $\text{gmult}(\lambda) = k$. We want to show that:

alg.mult$(\lambda) \geqslant k$

equivalently: $p_A(x) = (\lambda - x)^k \cdot f(x)$ for some polynomial $f(x)$

may or may not have a factor of $(\lambda - x)$

- Because $\text{gmult}(\lambda) = k$, we can find linearly independent eigenvectors

$$\vec{v}_1, \ldots, \vec{v}_k \in E_\lambda$$

- We can then extend this linearly independent set to a basis of $\mathbb{R}^n$:

$\mathcal{B} = \{\underbrace{\vec{v}_1, \ldots, \vec{v}_k}_{\text{eigenvects w/ eigenval } \lambda}, \vec{w}_{k+1}, \ldots, \vec{w}_n\}$ is a basis of $\mathbb{R}^n$, for some vectors $\vec{w}_{k+1}, \ldots, \vec{w}_n \in \mathbb{R}^n$

Set $S = [\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_{k+1}, \ldots, \vec{w}_n]$

- With respect to this basis, A looks like:

$$S^{-1}AS = \begin{bmatrix} [\underbrace{A\vec{v}_1}_{\lambda \vec{v}_1}]_{\mathcal{B}} & \cdots & [\underbrace{A\vec{v}_k}_{\lambda \vec{v}_k}]_{\mathcal{B}} & [\underbrace{A\vec{w}_{k+1}}_{?}]_{\mathcal{B}} & \cdots & [\underbrace{A\vec{w}_n}_{?}]_{\mathcal{B}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & \cdots & 0 & & & \\ 0 & \lambda & & \vdots & & & \\ \vdots & 0 & & 0 & [A\vec{w}_{k+1}]_{\mathcal{B}} & \cdots & [A\vec{w}_n]_{\mathcal{B}} \\ & \vdots & & \lambda & & & \\ & & & 0 & & & \\ 0 & 0 & & 0 & & & \end{bmatrix}$$

k cols

- Thus the characteristic polynomial of $S^{-1}AS$ is:

$$p_{S^{-1}AS}(x) = \det(S^{-1}AS - \lambda I_n) = \underbrace{(\lambda - x) \cdots (\lambda - x)}_{k \text{ copies}} \cdot (\text{some polynomial})$$

$$= (\lambda - x)^k \cdot f(x)$$

- But by last time, we know:

$$p_{S^{-1}AS}(x) = p_A(x)$$

QED

7.4 Why do we care about eigenvectors?

- They tell us about the geometry of our transform.

e.g. $T: \mathbb{R}^3 \to \mathbb{R}^3$ rotation around $\vec{v}$, then $\vec{v}$ is an eigenvector of T, $\lambda = 1$

e.g. $T: \mathbb{R}^3 \to \mathbb{R}^3$ reflection across a plane V, then

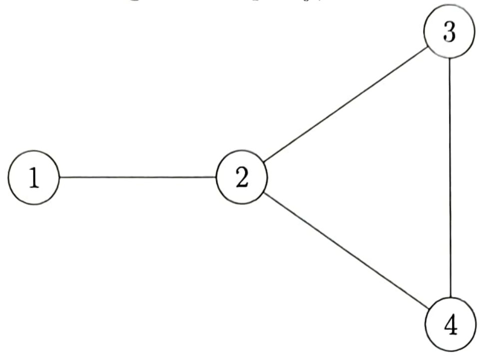
$V = E_1$ (eigenspace of $\lambda = 1$), $V^\perp = E_{-1}$

- Diagonalization: diagonal matrices are the nicest ones to work with.

- 7.4: understanding long-term behavior of a "dynamical system" (Markov chain)

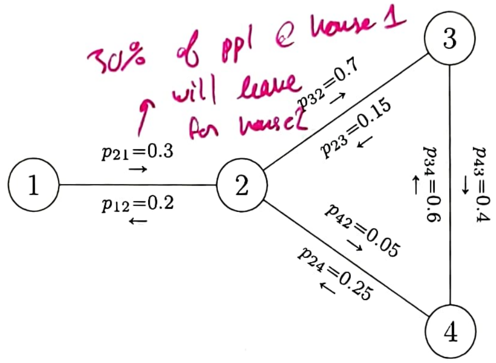
7.4 worksheet

Suppose there are four houses hosting a block party, connected to each other like so:



Every hour, some percent of people at one house leave and go to another house, as described by the following diagram:

30% of ppl @ house 1 will leave for house 2



Q Is there an "equilibrium" population for each house?

So for instance, the value of $p_{21} = 0.3$ means that house 2 will get 30% of the people in house 1 every hour. This process is an example of a *Markov process*, also called a *Markov chain*.

1. What proportion of people at house 1 will stay at house 1 after an hour? We call this number p_{11}. everyone in house 1: either stay @1, or go to 2. ⇝ $(1-0.3) = 70\%$ ppl @ 1 will stay
2. In general we let p_{ii} be the proportion of people in house i who decide to stay in house i when the hour changes. Find p_{22}, p_{33}, and p_{44} as well. $p_{22}: 1 - 0.2 - 0.05 - 0.7 = 0.05 = 5\%$
3. Let $x_i(t)$ be the number of people in house i, t hours after midnight. Let $\vec{x}(t)$ denote the vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

(this is an example of a *vector-valued function*: its input is a number t and its output is a vector $\vec{x}(t)$). Suppose that

$$\vec{x}(0) = \begin{bmatrix} 100 \\ 300 \\ 500 \\ 200 \end{bmatrix}$$

Find $x_2(1)$.

4. Find a matrix A such that $\vec{x}(t+1) = A\vec{x}(t)$. This is called the *transition matrix* of the markov process.