Math 214 7.3 worksheet

Facts and defs: Let A be an $n \times n$ matrix

- An *eigenvector* of A is a nonzero $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$.
- The function $p_A(x) = \det(A - xI_n)$ ~~$\neq 0$~~ is a polynomial in x, called the *characteristic polynomial* of A. A number $\lambda \in \mathbb{R}$ is an eigvenvalue of A if and only if $p_A(\lambda) = 0$. The *algebraic multiplicity* of λ is the biggest power of $(\lambda - x)$ dividing $p_A(x)$.
- The *eigenspace* of λ is $E_\lambda = \ker(A - \lambda I_n)$. This is the set of eigenvectors of A with eigenvalue $\cancel{0}$ λ (along with the zero vector, which doesn't count as an eigenvector).
- A is *diagonalizable* if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. If $\mathscr{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is such a basis, and $S = [\vec{v}_1, \dots, \vec{v}_n]$, then

$$\underbrace{S^{-1}AS}_{\text{matrix of the transform } T(\vec{x})=A\vec{x} \text{ with respect to } \mathscr{B}} = \begin{bmatrix} [A\vec{v}_1]_{\mathscr{B}} & \cdots & [A\vec{v}_n]_{\mathscr{B}} \end{bmatrix} = \begin{bmatrix} [\lambda_1\vec{v}_1]_{\mathscr{B}} & \cdots & [\lambda_n\vec{v}_n]_{\mathscr{B}} \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{diagonal matrix w/ eigenvalues of } A \text{ along the diagonal}}$$

If λ is a multiple root of $p_A(\lambda)$, does that corresp. to multiple eigenvectors of A?

A: sometimes.

1. Find the eigenvectors of the following matrices. Are they diagonalizable? What are the algebraic multiplicites of the eigenvalues?

(a) $\begin{bmatrix} 0 & -2 & 2 \\ -3 & -2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$ Hint $p_A(x) = -x^3 + 16x$

Ans $\lambda = -4, 4, 0$

vects: $\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$

Diagonalizable? Yes

(b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$p_A(\lambda) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (2-\lambda)(2-\lambda)(3-\lambda) = 0$$

eigenval of 2, w/ alg. mult = 2

eigval of 3 w/ alg. mult 1.

Eigenvects: $E_2 = \ker(A - 2I_3) = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ ← eigenvector for $\lambda = 2$

$E_3 = \ker(A - 3I_3) = \ker\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ ← eigenvect for $\lambda = 3$

(c) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

$p_A(\lambda) = \det\begin{pmatrix} 1-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \\ -1 & -1 & 2-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda)(2-\lambda) = 0$

$\lambda = 2$, alg mult 2 $\qquad$ $\lambda = 1$, alg. mult 1.

Is diagonalizable!

$E_2 = \ker(A - 2I_3) = \ker\begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} = \operatorname{span}\underbrace{\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}}_{\text{2 lin. indep eigenvects for } \lambda = 2}$

$E_1 = \ker(A - I_3) = \ker\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ → 1 eigenvect for $\lambda = 1$

λ has geometric mult. 2

Definition: the *geometric* multiplicty of an eigenvalue λ is $\dim(E_\lambda) = \dim(\ker(A - \lambda I))$

Fact: a matrix is diagonalizable if and only if $\sum_{\lambda \text{ is an eigenval of } A} \dim(E_\lambda) = n$, $\sum$ geom. mult(λ) $= n$

Fact: the geometric multiplicity of an eigenvalue is always $\leq$ than its algebraic multiplicity.

In other words, each additional (linearly independent) eigenvector for λ gives an additional factor of $(\lambda - x)$ dividing $p_A(x)$

e.g. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin indep eigenvects

eigenvals $2 \quad 2 \quad -4$

$\rightsquigarrow p_A(x) = (2-x)(2-x)(-4-x) \cdot f(x)$

may or may not be divisible by $(2-x)$ or $(-4-x)$

Let's try to prove this fact!

2. Find the determinant of the following matrix (it's not as bad as it looks!)

$$\det\begin{pmatrix}\begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 2 & 0 & 6 & 9 \\ 0 & 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}\end{pmatrix}$$

expand along left-most columns

$= 1 \cdot \det\begin{pmatrix} 2 & \\ 0 & \square \\ 0 & \\ 0 & \end{pmatrix} = 1 \cdot 2 \cdot \det\begin{pmatrix} 3 & \\ 0 & \square \\ 0 & \end{pmatrix}$

$= 1 \cdot 2 \cdot 3 \cdot \det\begin{pmatrix} 4 & 1 \\ 3 & 0 \end{pmatrix}$

$= 6 \cdot -3 = -18$

In general:

$$\det \begin{pmatrix} \lambda_1 & & 0 & \\ & \ddots & & B \\ 0 & & \lambda_k & \\ & 0 & & A \end{pmatrix} = \lambda_1 \cdots \lambda_k \det(A)$$

$k \times (n-k)$ matrix

$k \times k$ matrix

n

"Block matrix"

(See thm 6.1.5)

3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let S be an invertible matrix. What is $\det(S^{-1}AS)$?

$$\det(S^{-1}AS) = \boxed{\det(S^{-1})} \det(A) \det(S) = \det(A)$$

$\frac{1}{\det(S)}$

$$= 1 \cdot 4 - 2 \cdot 3 = -2$$

4. Let A be a square matrix and $B = S^{-1}AS$. Show that $p_A(x) = p_B(x)$.

Want to show: $\det(A - xI_n) = \det(B - xI_n)$

$$\det(B - xI_n) = \det(S^{-1}AS - xI_n)$$

$$\det(S^{-1}(A - xI_n)S) = \det(S^{-1}AS - S^{-1}(xI_n)S)$$

check: $(xI_n)S = SxI_n$

$$= \det(A - xI_n)$$

by prob 3 idea.

Now let's prove the theorem!

Theorem If A is a square matrix and λ is an eigenvalue of A, then $\mathrm{gmult}(\lambda) \leqslant \mathrm{algmult}(\lambda)$

Proof