Exam 3 practice worksheet

You do not need to turn in this worksheet!

1 Verifying Trig IDs (Section 5.2 of the book)

There isn't an algorithm for verifying trig identities, so these problems take creativity, intuition, and persistence. There are a few guiding principles, however:

1.1 Start problems off by picking the low-hanging fruit

If you see an immediate application of a trig ID you know, it's a good idea to go ahead and apply it. For instance, if you see $1 - \sin^2 x$ somewhere, chances are that you want to convert it to $\cos^2 x$. Likewise, if you come across $\sec x \cdot \cos x$, you'll probably want to replace that with a 1. Here are some problems that illustrate this idea:

1. Show that $\cot^2 x \left(\sec^2 x - 1\right) = 1$
2. Show that $\dfrac{\tan x + \cot x}{\tan x \cot x} = \tan x + \cot x$
3. Show that $\cos^2 x + \cos^2 \left(\dfrac{\pi}{2} - x\right) = 1$

1.2 Algebraic simplification

After you've checked for low-hanging fruit, the next step is often to do some Math 1010-style simplification—add fractions, factor expressions, simplify expressions, and so on. Or maybe you need This requires algebra skills more than trig skills, so make sure you're comfortable with the algebra! Here are a few practice algebra problems:

4. Write $x + \dfrac{1}{x}$ as one fraction
5. Simplify
$$\frac{x^2 - 4}{x + 2}$$
6. Simplify
$$\frac{\left(\dfrac{x}{x+2}\right)}{\left(\dfrac{3}{x+2}\right)}$$
7. Simplify
$$\frac{\left(\dfrac{x}{x+2}\right)}{\left(\dfrac{x+2}{3}\right)}$$

And here are some problems that require simplification

8. Show that $\cos x - \dfrac{1}{\cos x} = \dfrac{-\sin^2 x}{\cos x}$
9. Show that $\dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} = \dfrac{2}{\cos x}$
10. Show that $\sin^2 x - \sin^4 x = \cos^2 x - \cos^4 x$

1.3 Brute force it

When all else fails, you can convert everything to sins and coss and go from there. This approach will work for every problem, but it could get really messy.

11. Solve problem 1 by converting everything to sins and coss first. Notice how much longer the solution is now than it was before.
12. Show that $\cos x - \sec x = -\sin^2 x \sec x$

1.4 Further practice problems

Of course, on the exam (and whenever else you need to verify trig identities), you won't be told which method to use. Here are some more problems without any hints:

13. Show that $(1 + \sin x)(1 - \sin x) = \cos^2 x$
14. Show that $\dfrac{\tan x + \cot x}{\sin x \cos x} = \sec^2 x + \csc^2 x$
15. Show that $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$
16. Show that $\sin x \cdot \csc\left(\dfrac{\pi}{2} - x\right) = \tan x$

2 Solving trig equations (Section 5.3 of the book)

This section is really the meat of chapter 5.

2.1 Introductory examples

17. Solve for x: $2\sin(x) = 1$
18. Solve for x: $2\cos(x) + \sqrt{3} = 0$
19. Solve for x: $2\cos(2x) + \sqrt{3} = 0$
20. Solve for x: $\cot(2x) = \sqrt{3}$

2.2 Factoring expressions

The goal of each method we have for solving trig equations is to reduce the problem to something like $\sin x = 1/2$. One way to do this is by factoring. We know from algebra that if $f(x) \cdot g(x) = 0$ at $x = x_0$, then $f(x_0) = 0$ or $g(x_0) = 0$ (and vice-versa). Thus the solutions to something like $(\sin x - 1) \cdot \left(\cos x - \dfrac{1}{2}\right) = 0$ are the solutions to $\sin x - 1 = 0$ and $\cos x - \dfrac{1}{2} = 0$. **NOTE:** this only works when we're setting things to zero!! For example $x = 30°$ is a solution to $\cos(x) \cdot \sin(x) = \dfrac{\sqrt{3}}{4}$, but neither $\cos(30°)$ nor $\sin(30°)$ equals $\dfrac{\sqrt{3}}{4}$.

21. Solve $\sin x + \tan x = 0$
22. Solve $\sin x \cos x + \dfrac{1}{2}\cos x + \dfrac{1}{2}\sin x + \dfrac{1}{4} = 0$
23. Solve $\sin^2 x + 2\sin x + 1 = 0$

A note on grammar: one would say that the *set of solutions* for this equation is $x = \ldots$ *and* $x = \ldots$. However, one would say that x *is a solution if* $x = \ldots$ *or* $x = \ldots$. I don't care whether you use "and" or "or".

3 Sum and difference forumlae (section 5.4 of the book)

24. What's $\sin(45°)$?
25. What's $\sin(90°)$?
26. What's $\sin(135°)$?
27. 17th century mathematician Sir Isaac Newton once famously said "I'm pretty sure that $\sin(a+b) = \sin(a) + \sin(b)$"[1]. Write two sentences about why Sir Isaac Newton was wrong.
28. Compute $\cos(37°)\cos(8°) - \sin(37°)\sin(8°)$
29. Compute $\tan(105°)$
30. Compute $\sin(15°)$

3.1 Using sum and difference formulas to solve trig equations

31. Solve $\sin(x+\pi) - \sin x + 1 = 0$
32. Solve $\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{7\pi}{6}\right) = \frac{\sqrt{3}}{2}$ for x

4 Double angle, power reduction, half angle (section 5.5 of the book)

Solving multiple-angle trig equations.

33. What's $\cos\left(\frac{\pi}{4}\right)$?
34. What's $\cos\left(\frac{\pi}{2}\right)$?
35. In 1931, at age 25, Austrian mathematician Kurt Gödel shocked the mathematical community by saying "$\cos(2x) = 2\cos x$ for all x"[2]. Is this true?

4.1 Solving equations using multiple angle formulas

Hint: a lot of these problems involve factoring. See section 2.2

36. Solve for x: $\cos(2x) = 2\cos(x)$
37. Solve for x: $\sin 2x = \sin x$
38. Solve for x: $\tan 2x - \cot x = 0$

4.2 Reducing powers

39. Write $\sin^2 x \cos^2 x$ as an expression in terms of the first power of cosine.

5 Law of Sines (Section 6.1 of the book)

The point of the law of sines is to help you solve triangles that are not right triangles. In every problem below, A is the angle opposite side a, B is the angle opposite side b, and C is the angle opposite side c.

[1] He never actually said this

[2] Actually, he said the much more shocking fact that some statements are neither provably true nor provably false

5.1 Two angles and a side are given

Whenever you're given two angles, the first step is to find the missing angle.

40. Solve the triangle where $A = 56°$, $B = 10°$ and $c = 5$. Use the table of sine values at the end of the worksheet.

41. Solve the triangle where $A = 10$, $B = 114$, and $a = 6$

5.2 Side-Side-Angle triangles

The other time the law of sines is useful is when you're given two sides of a triangle and an angle not between them. Things can get weird in this case—sometimes there won't be a solution, and sometimes there will be two solutions. Here's an example where there are no solutions: if $A = \arcsin(0.4)$, $a = 3$, and $b = 10$, then the law of sines tells us that $\sin B = 4/3$. But $4/3$ is greater than 1, so this is impossible.

For the following problems, determine if a solution exists.

42. $A = \arcsin(0.9)$, $a = 1$, $b = 2$.

43. $B = \arcsin(0.4)$, $c = 3$, $b = 2$.

44. $C = \arcsin(0.3)$, $a = 100$, $c = 4$.

If a solution does exist, you find out if there are one or two solutions as follows: suppose now you're given $A = 10$, $a = 3$, and $b = 15$. Then using our table of sine values we see that $\sin A = 1/6$, so $\sin B = b \cdot \frac{\sin A}{a} = 5/6$. Using our table of sine values again, we see that $\sin(56°) = 5/6$. Because of this, and the fact that $56°$ is between $-90°$ and $90°$, we conclude that $\sin^{-1}(5/6) = 56°$. Now, we know that B is less than 180 degrees and its sine is $5/6$. It turns out that this gives two options for that B could be—either $b = \arcsin(5/6) = 56°$, or $b = 180 - \arcsin(5/6) = 124°$. We also know that $A + B < 180°$ since A and B are in the same triangle. Does this happen for both of the answers we got for B? If $B = 56°$, then $A + B = 10° + 56° = 66° < 180°$, and if $B = 124°$, then $A + B = 10° + 124° = 134° < 180°$. So there are two answers for this problem. That is, there are two triangles with these measurements.

For the following problems, determine how many solutions exist:

45. $B = 24°$, $b = 4$, $c = 2$

46. $A = 114°$, $c = 6$, $a = 2$

47. $C = 56°$, $c = 1$, $a = 2$

6 Law of Cosines (Section 6.2 of the book)

The law of sines is great, but it doesn't always help! Especially if we're given three sides of a triangle and want to find its angles, or if we're given two sides of a triangle and the angle between them. Using the law of sines and law of cosines, you can solve any triangle. When using the law of cosines, it's helpful to remember the following tips:

- You never need to apply the law of cosines more than once for any problem
- If a triangle has an obtuse angle, it must be the largest angle of that triangle

48. Suppose a triangle has side lengths $a = 2$, $b = 3$, $c = 4$. Find the angle C.

49. Suppose a triangle has measurements $A = 34$, $b = 5$, $c = 10$. Find side a.

For the following, say whether or not you need the law of cosines to solve the triangle:

50. $A = 40$, $a = 5$, $b = 6$

51. $A = 40$, $B = 20$, $a = 10$

52. $A = 40$, $b = 5$, $c = 6$

53. $a = 5$, $b = 6$, $c = 7$

7 Table of sine values

θ (degrees)	$\sin(\theta)$
8	1/7
10	1/6
12	0.2
24	0.4
25	5/12
56	5/6
114	12/13

8 Table of cosine values

θ (degrees)	$\cos(\theta)$
104°	-0.25
34°	5/6