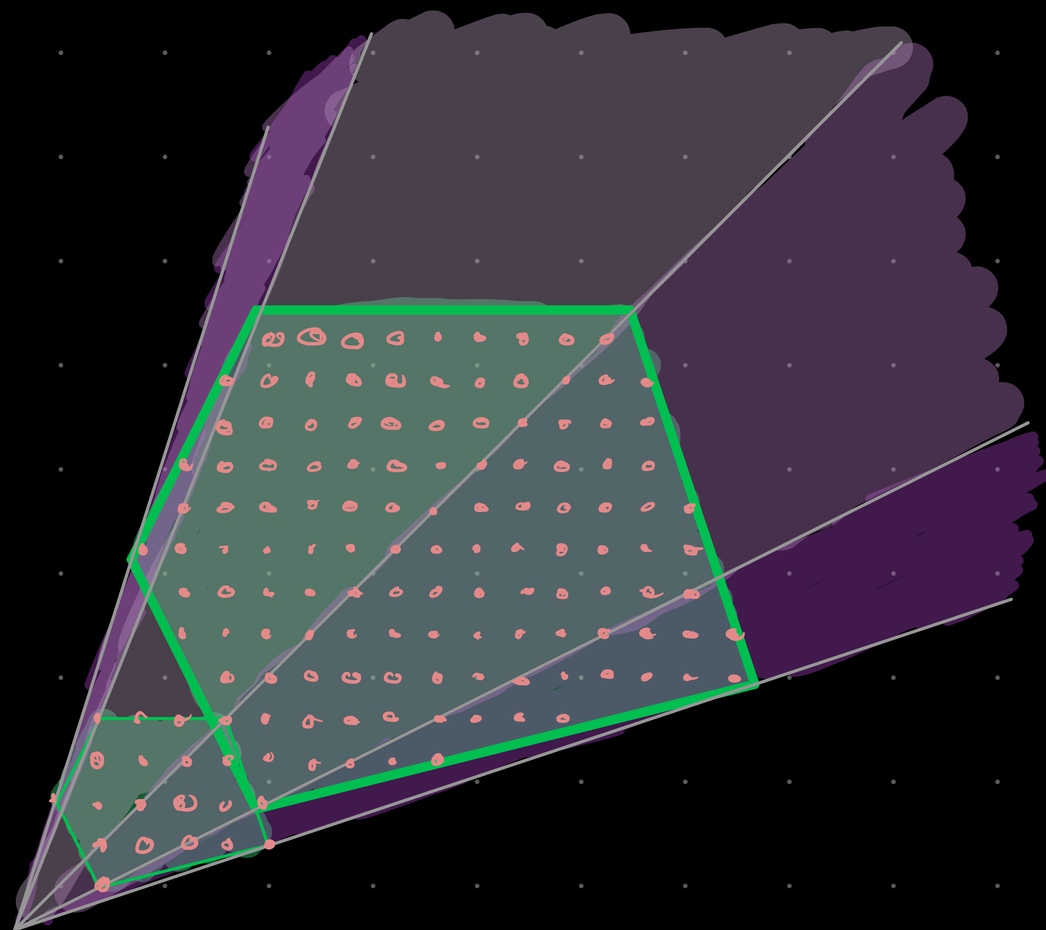
Counting Points in Polytopes

UM math club

March 31, 2022



= keep, = write during talk.

Draw a shape in $\mathbb{R}^2$

Q Can we predict area of P from # grid pts in P, and vice-versa?

↳ elements of $\mathbb{Z}^2 \cap P$

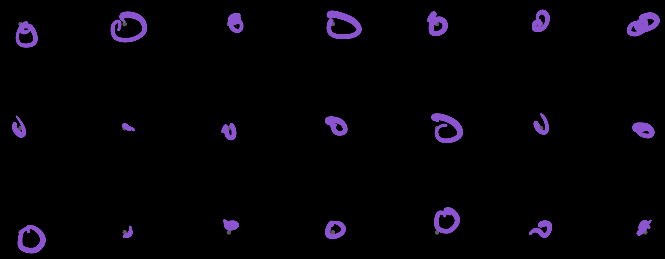
$\mathbb{Z}^2 = \{(m,n) \mid m,n \text{ integers}\}$

"integer lattice"

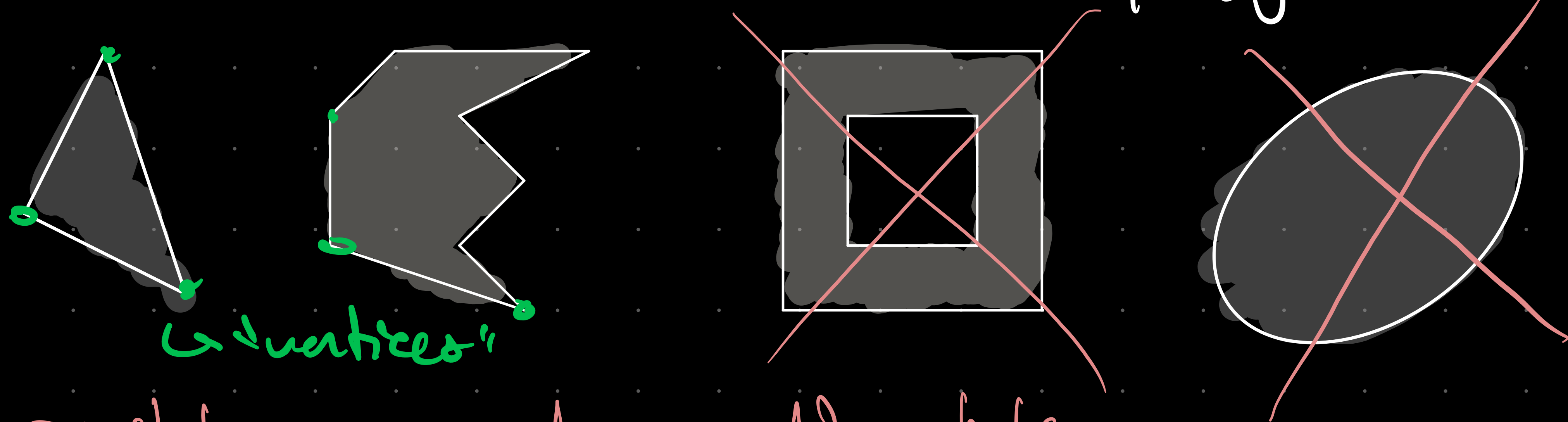
<u>Part I</u> : 2D situation.

- **Problem** : there are some wild shapes out there!

eg.

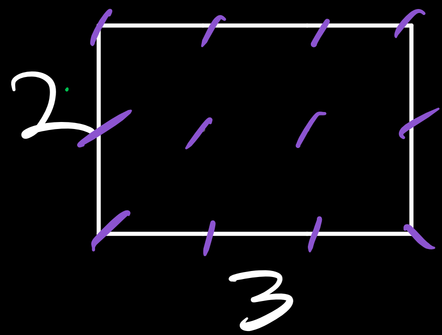


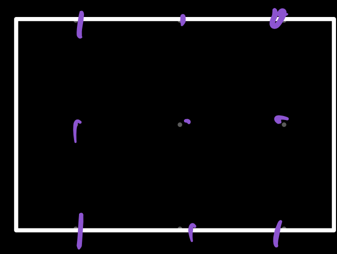
- Let's restrict ourselves to polygons

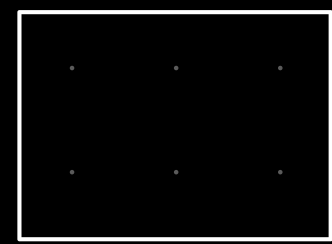


Finitely many sides.   No holes.

e.g. rectangles







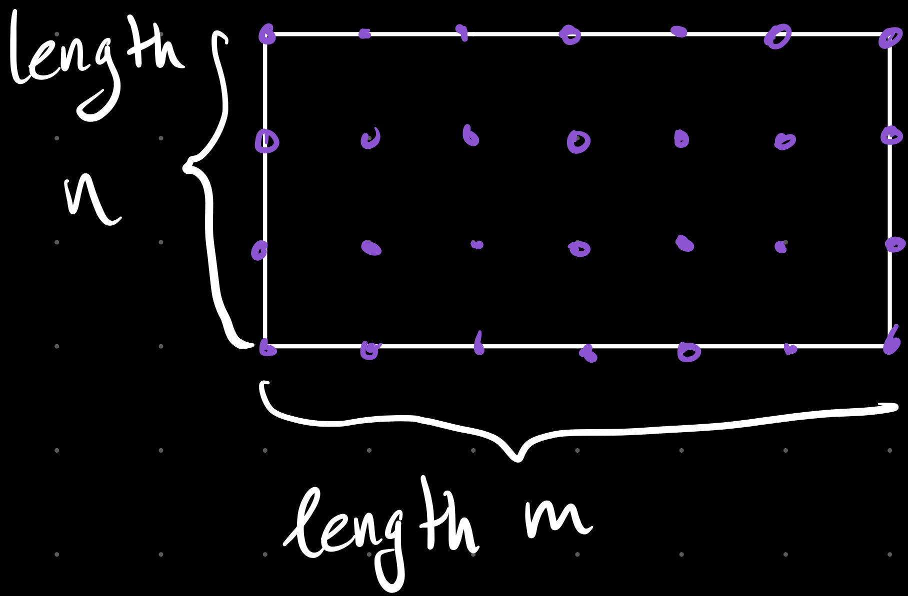
$A = 6$, #pts $= 12$

$A = 6$, #pts $= 9$

#pts $= 6$

We restrict ourselves further to lattice polygons (vertices are grid pts)

Going back to rectangles



Area $= mn$

$\#\text{pts} = (m+1)(n+1) > mn$

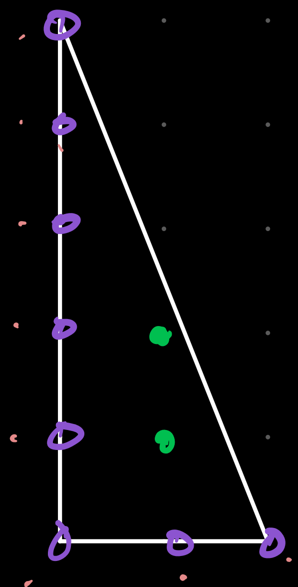
$$\#\text{pts} = \underbrace{mn}_{\text{area}} + \underbrace{m+n}_{\frac{\#\text{pts in ``boundary''}}{2}} + 1$$

Let $I = \#$ "interior pts"; $B = \#$ "boundary pts"

$\#\text{pts} = I + B = A + \frac{B}{2} + 1$

$\rightsquigarrow \quad A = I + B/2 - 1$

$A = I + \frac{B}{2} - 1$?

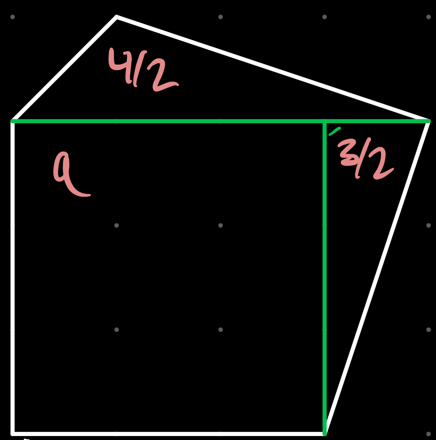


$$\text{Area} = \frac{2 \cdot 5}{2} = 5$$

$I = 2$

$B = 8$

$$5 = 2 + \frac{8}{4} - 1 \quad ✓$$



$$\text{Area} = 9 + \frac{3}{2} + 2 = 12.5$$

$I = 9$

$B = 9$

$$12.5 = 9 + \frac{9}{2} - 1 \quad ✓$$

Pick's Thm If P is a lattice polygon, and

I = # interior grid pts of P,

B = # boundary grid pts of P,

A = area of P,

then $A = I + \frac{B}{2} - 1$

"continuous" data

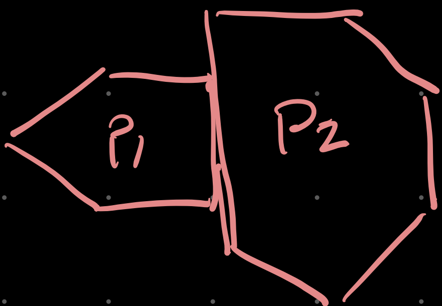
"discrete data"

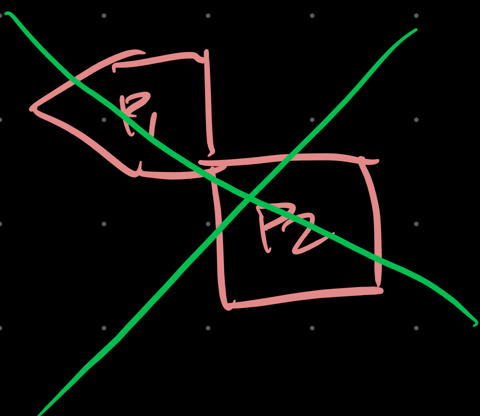
We've shown this for rectangles with horizontal and vertical sides

Let's prove it in general!

$$A = I + \frac{B}{2} - 1 \ ?$$

Claim 1: if P_1 and P_2 are lattice polygons satisfying Pick's thm, then $P_1 \cup P_2$ satisfies Pick's thm.





Pf Set $A_1 = \text{area}(P_1)$

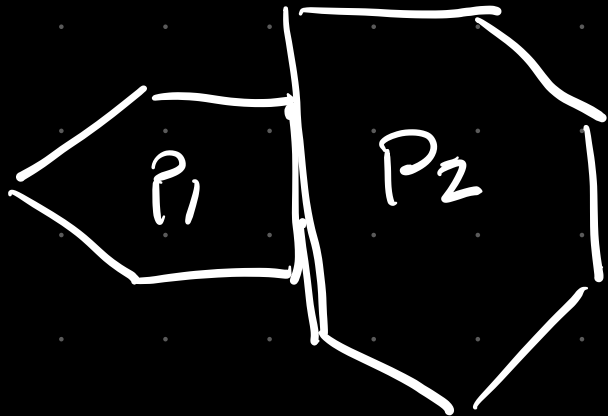
$I_1 = \#$ interior pts of P_1

etc.

$$\text{area}(P_1 \cup P_2) = \text{area}(P_1) + \text{area}(P_2)$$

$$= I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1$$

Claim: $$= I_{P_1 \cup P_2} + \frac{B_{P_1 \cup P_2}}{2} - 1$$



$$\text{area}(P_1 \cup P_2) = \text{area}(P_1) + \text{area}(P_2)$$

$$= I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1$$

Claim

$$= I_{P_1 \cup P_2} + \frac{B_{P_1 \cup P_2}}{2} - 1$$

Let $E = \#$ grid pts in $P_1 \cap P_2$

$$\Rightarrow I_{P_1 \cup P_2} = I_1 + I_2 + E - 2$$

need lattice property!!

$$B_{P_1 \cup P_2} = B_1 + B_2 - 2E + 2$$

$$\Rightarrow I_{P_1 \cup P_2} + \frac{B_{P_1 \cup P_2}}{2} - 1 =$$

$$I_1 + I_2 + \cancel{E} - 2 + \frac{B_1 + B_2 \cancel{- 2E} \cancel{+ 2}}{\cancel{2}} \cancel{-1}$$

$$= I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1 = \text{area}(P_1 \cup P_2) \checkmark$$

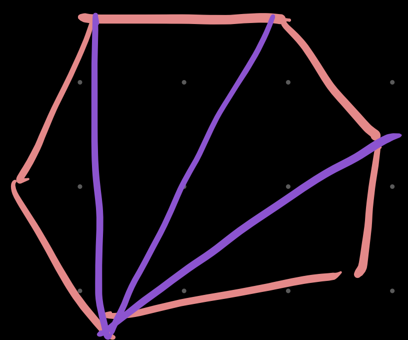
Rmk Same proof shows: if $P_1 \cup P_2$ satisfies Pick, and P_1 satisfies Pick, then P_2 satisfies Pick

$$\underbrace{I_1 + \frac{B_1}{2} - 1}_{\text{area}(P_1)} + I_2 + \frac{B_2}{2} - 1 = \underbrace{I_{P_1 \cup P_2} + \frac{B_{P_1 \cup P_2}}{2} - 1}_{\text{Area}(P_1 \cup P_2)}$$

$\text{area}(P_2) = \text{area}(P_1 \cup P_2) - \text{area}(P_1)$

Claim 2 any convex polygon has a "triangulation"

"two ears theorem"



$P = T_1 \cup \cdots \cup T_n$, T_i triangles

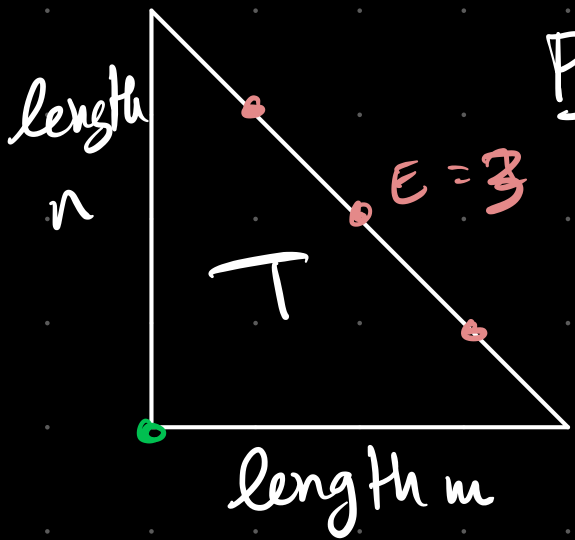
vertices of $T_i \subseteq$ vertices of P

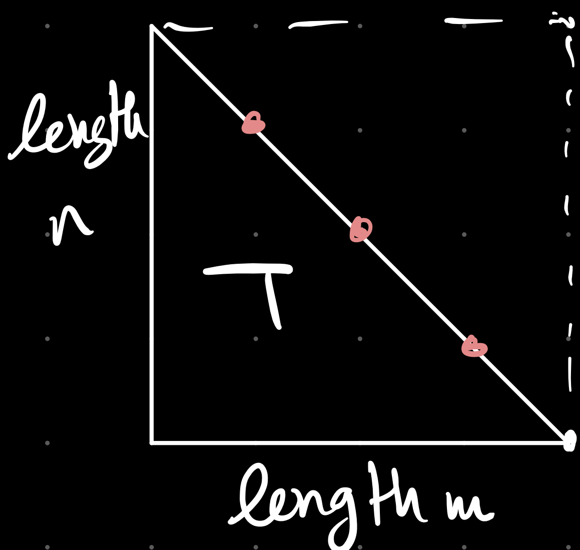
$T_i \cap T_{i+1} =$ a side of T_i

It suffices to prove Pick for Δs!

Claim Pick holds for lattice triangles

Case 1 Right triangle with a vertical leg





Pf Let E = # grid pts on hypotenuse, not counting vertices

$$B = m+1 + n+1 + E - 1 = m+n+E+1$$

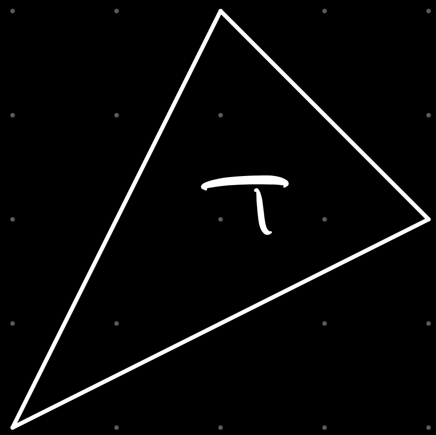
$$I = \frac{1}{2}\left(I_{square} - E\right)$$

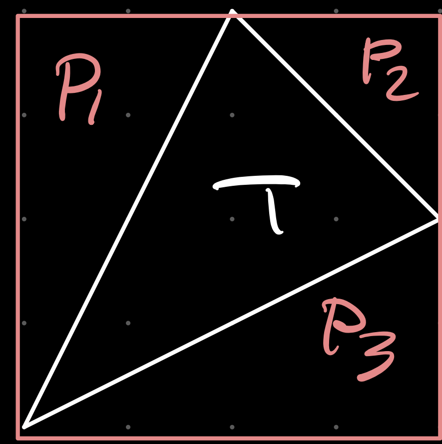
$$= \frac{1}{2}\left[(m-1)(n-1) - E\right]$$

Algebra: $I + \frac{B}{2} - 1 = \frac{mn}{2}$

$= \text{area}$

Case 2 any lattice triangle





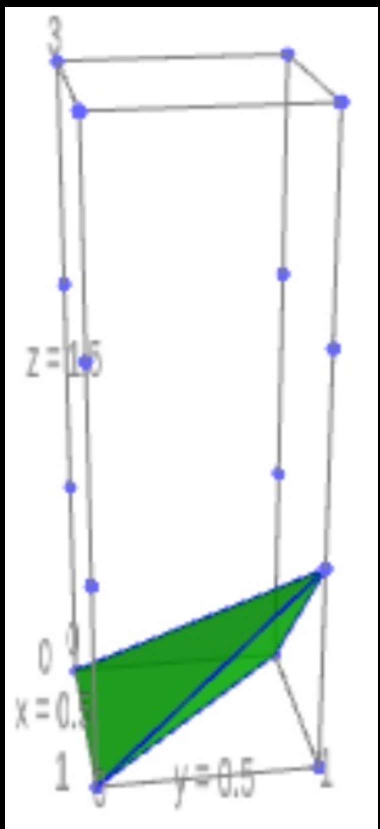
Pick holds for rectangle and for P_1, P_2, P_3

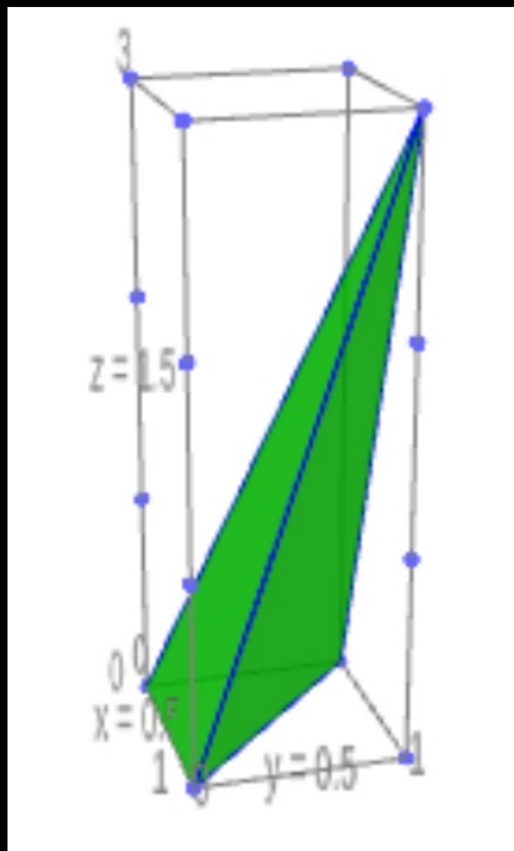
$\Rightarrow$ Pick holds for T! $\blacksquare$

Q Does Pick's thm generalize to higher dimensions?

A: not quite.

E.g. "Reeve's tetrahedra"





vertices: $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,1,r)$, $r \in \mathbb{N}$.

Same lattice pt counts,

Volume $\to \infty$ as $r \to \infty$.

But! Pick's theorem does generalize, in the following sense.

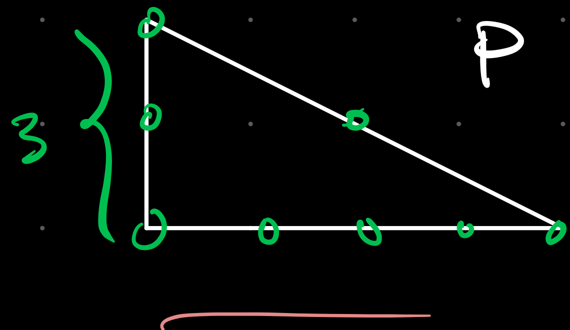
$$\#(\mathbb{Z}^2 \cap P) = \text{Total \# pts in } P = \overset{= A_P - \frac{B_P}{2} + 1}{I_P} + B_P = A_P + \frac{B_P}{2} + 1$$

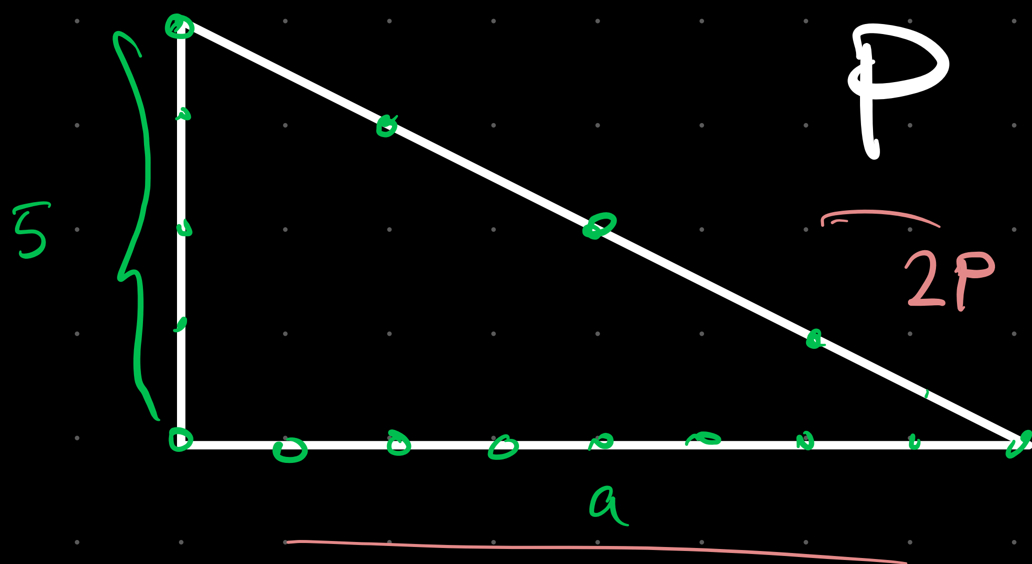
Enlarge P by a factor of $t \in \mathbb{N}$:

$$\#(\mathbb{Z}^2 \cap tP) = A_{tP} + \frac{B_{tP}}{2} + 1 \underset{\text{lemma}}{=} A_P t^2 + \frac{B_P}{2} t + 1$$

Check: $B_P = 8$

$B_{2P} = 16$





We've shown:

$\#(\mathbb{Z}^2 \cap tP) = \#$ grid pts in tP

$= A_P t^2 + \frac{B_P}{2} t + 1 \quad \forall t \in \mathbb{N}.$

degree-2 polynomial!

<u>Thm</u> (Ehrhart): Let P be a "<u>d-dimensional polygon</u>" in $\mathbb{R}^d$. Then there is a degree-d polynomial $f(t) = a_d t^d + \cdots + a_0$ s.t. $\#(\mathbb{Z}^d \cap tP) = f(t)$ for all $t \in \mathbb{N}$.

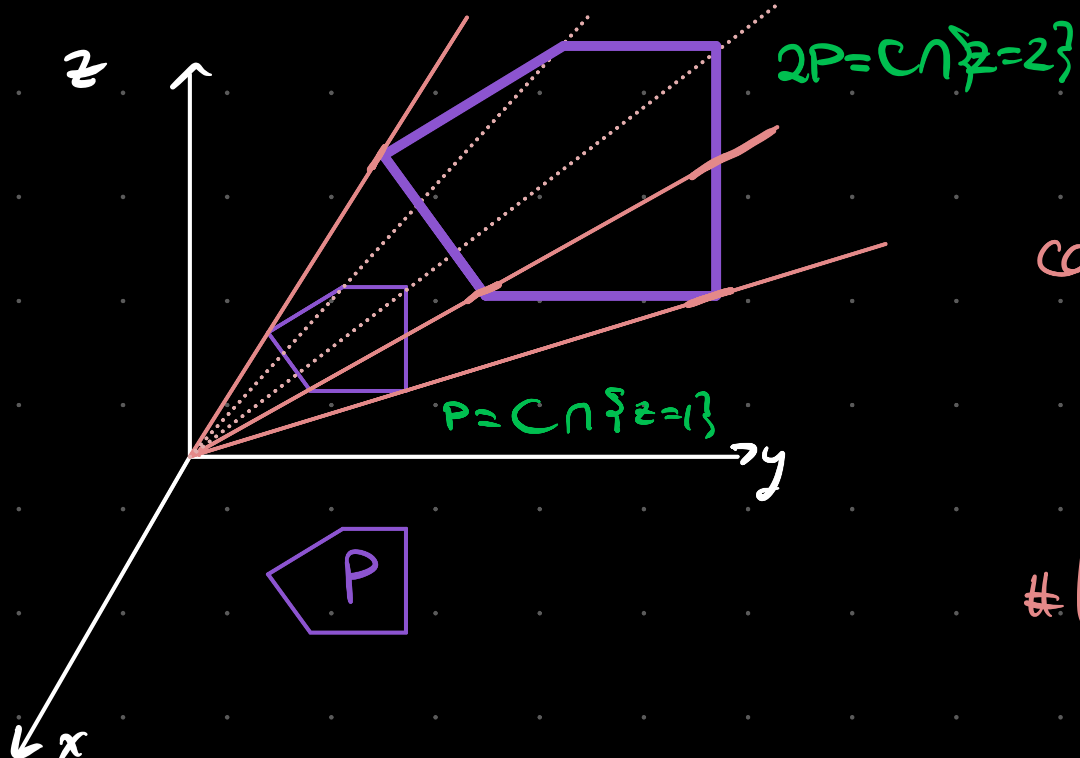
"polytope"

"Ehrhart polynomial"

Part II : Sketch of the proof in dim > 2

The proof uses lots of neat ideas from combinatorics !

Idea 1 : "coning over a polytope"



Def given $P \subseteq \mathbb{R}^d$,

$$\mathrm{cone}(P) = \left\{ (r\vec{x}, r) \in \mathbb{R}^{d+1} \;\middle|\; \begin{array}{l} \vec{x} \in P, \\ r \in \mathbb{R}_{\geq 0} \end{array} \right\}$$

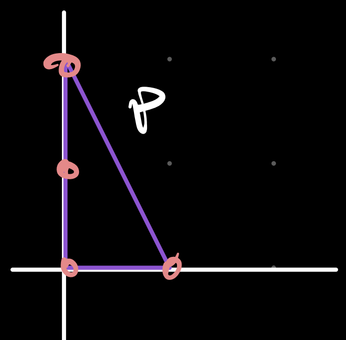
$$\#(\mathbb{Z}^d \cap tP) = \#(\mathrm{cone}(P) \cap \mathbb{Z}^d \times \{t\})$$

Idea 2 generating functions.

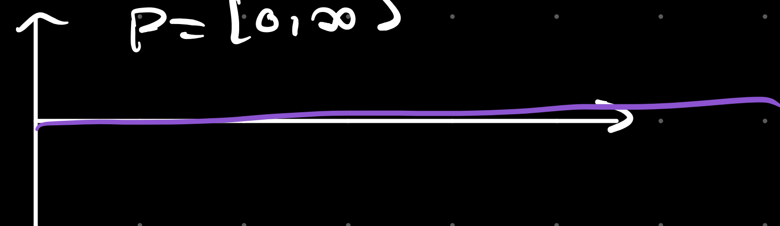
Given $P \subseteq \mathbb{R}^d$, consider

$$\sigma_P(z_1, \dots, z_d) = \sum_{(a_1, \dots, a_d) \in \mathbb{Z}^d \cap P} z_1^{a_1} \cdots z_d^{a_d}$$

e.g.



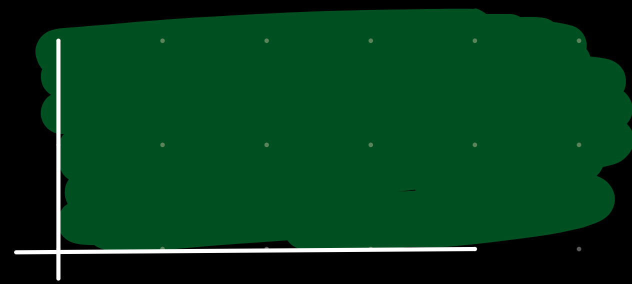
$P = [0, \infty)$



$$\sigma_P(z_1, z_2) = z_1^0 z_2^0 + z_1^1 z_2^0 + z_1^0 z_2^1 + z_1^0 z_2^2$$

$$\sigma_P(z_1, z_2) = \sum_{n \geq 0}^{\infty} z_1^n = \frac{1}{1 - z_1}$$

e.g. $P = \{(x,y) \mid x,y \geq 0\}$,

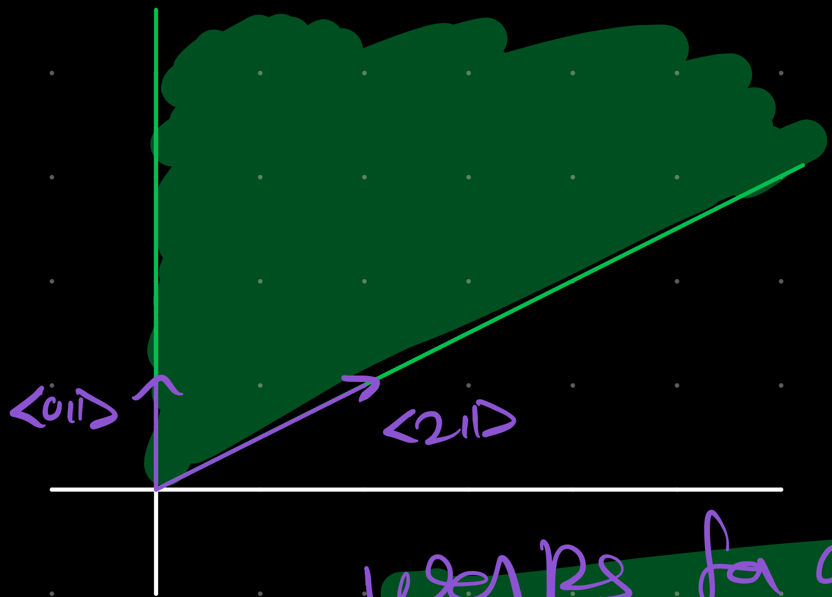


$\mathbb{Z}^2 \cap P = \{(i,j) \mid i,j \in \mathbb{N}\}$

$$\sigma_P(z_1,z_2) = \sum_{i=0,\, j=0}^{\infty} x^i y^j = \left(\sum x^i\right)\left(\sum y^j\right)$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-y}$$

eg $P = \{\lambda_1\langle 0,1\rangle + \lambda_2\langle 2,1\rangle \mid \lambda_1,\lambda_2 \geq 0\}$



Each λ_i = integer + $\overset{[0,1)}{\mu_i}$

Set $\Pi =$

$\{\mu_1\langle 0,1\rangle + \mu_2\langle 2,1\rangle \mid 0 \leq \mu_i < 1\}$

works for any cone!!!

$\forall x \in P$: $x = i\cdot\langle 0,1\rangle + j\cdot\langle 2,1\rangle + \alpha$ $\left(\begin{array}{l} i,j \in \mathbb{Z}, \\ \alpha \in \Pi \end{array}\right)$

$x \in \mathbb{Z}^2 \iff \alpha \in \mathbb{Z}^2$

$$\Rightarrow \sigma_P(z_1,z_2) = \sum_{\alpha\in\mathbb{Z}^2\cap\Pi} (x^0y^1)^i (x^2y)^j x^{\alpha_1}y^{\alpha_2}$$

$$= \frac{1}{1-y}\cdot\frac{1}{1-x^2y}\cdot\sigma_\Pi(x,y) = \frac{1+xy}{(1-y)(1-x^2y)}$$

Let $P \subseteq \mathbb{R}^d$, $C = \operatorname{cone}(P) \subseteq \mathbb{R}^{d+1}$

Then $$\sigma_C(z_1, \ldots, z_{d+1}) = \sum_{(a_1, \ldots, a_{d+1}) \in \mathbb{Z}^{d+1} \cap C} z_1^{a_1} \cdot \ldots \cdot z_{d+1}^{a_{d+1}}$$

$$= \sum_{\substack{a_{d+1} \in \mathbb{N} \\ (a_1, \ldots, a_d) \in a_{d+1} \cdot P \cap \mathbb{Z}^d}} z_1^{a_1} \cdots z_{d+1}^{a_{d+1}}$$

terms where $a_{d+1} = 2$
$= \#(\mathbb{Z}^d \cap 2P)$

terms where $a_{d+1} = 1$
$= \# \mathbb{Z}^d \cap P$

$\Rightarrow \sigma_C(1, \ldots, 1, z) = c_0 + c_1 z + c_2 z^2 + \cdots,$

where $c_n = \#(\mathbb{Z}^d \cap nP)$

Finale

$P \subseteq \mathbb{R}^d$ a polytope

$C = \mathrm{cone}(P)$. $\Rightarrow \sigma_C(1, \dots, 1, z) = \sum_{n=0}^{\infty} (\# \mathbb{Z}^d \cap nP) z^n$

Technical lemma 1: as in earlier example,

$$\sigma_C(z_1, \dots, z_{d+1}) = \frac{\sigma_{\Pi}(z_1, \dots, z_{d+1})}{(1 - z_1^{a_{11}} \cdots z_{d+1}) \cdots (1 - z_1^{a_{k1}} \cdots z_{d+1})}$$

$\longrightarrow$ finite sum

$$\Rightarrow \sigma_C(1, \dots, 1, z_{d+1}) = \frac{\sigma_{\Pi}(1, \dots, 1, z)}{(1-z)^k}$$

Finale

$P \subseteq \mathbb{R}^d$ a polytope

$$C = \operatorname{cone}(P). \Rightarrow \sigma_C(1,\dots,1,z) = \sum_{n=0}^{\infty} \left(\#\mathbb{Z}^d \cap nP\right) z^n$$

$$= \frac{\sigma_{\Pi}(1,\dots,1,z)}{(1-z)^R}$$

polynomial

Technical lemma 2:

polynomial, deg $\leq d$ w/ $g(1) \neq 0$

If $$\sum_{n \geq 0}^{\infty} f(n)\, z^n = \frac{g(z)}{(1-z)^{d+1}}$$

Then $f(n)$ is a polynomial function!

References

Questions ?

Q How do we define a "polytope"?

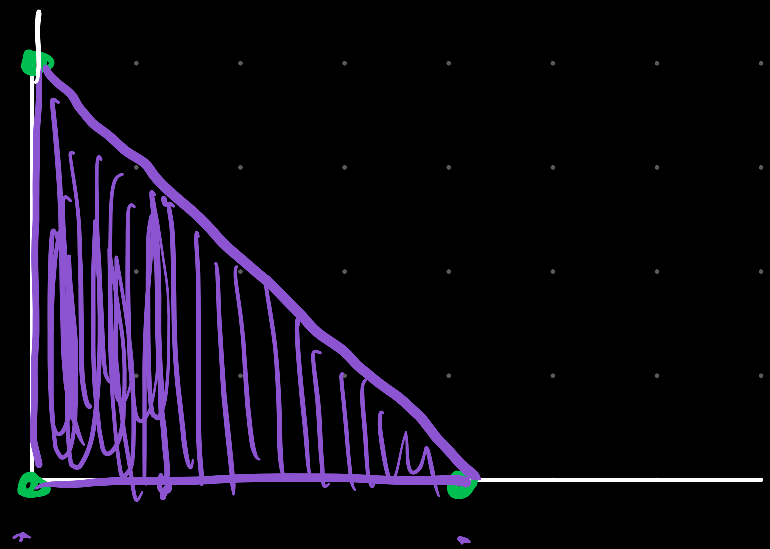
Def A set $S \subseteq \mathbb{R}^d$ is called **convex** if $\forall p, q \in S$, the line segment p —— q is $\subseteq S$.

$\{tp + (1-t)q \mid t \in [0,1]\}$.

Def The **convex hull** of a set $S \subseteq \mathbb{R}^d$ is the smallest convex set containing S $=$ $\text{conv}(S)$

$$= \bigcap_{\substack{C \supseteq S \\ \text{convex}}} C$$

e.g. Convex hull of $\{(0,0), (4,0), (0,4)\}$?



= triangle with vertices $(0,0)$, $(4,0)$, $(0,4)$.

Fact any convex polygon is the convex hull of its vertices.

Same with 3D polyhedra

Ices $\quad S = \mathrm{conv}\{\vec{v}_1, \ldots, \vec{v}_{12}\}$

Def A <u>convex polytope</u> is (a set which equals $\mathrm{conv}\{\vec{v}_1, \dots, \vec{v}_n\}$ for some $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^d$)

Def ("lattice polytope" if $\vec{v}_i \in \mathbb{Z}^d \ \forall i$)

Thm if $P \subseteq \mathbb{R}^d$ is a convex lattice polytope, then $\exists$ degree-d polynomial $f(t)$ s.t.

$$\#(\mathbb{Z}^d \cap tP) = f(t) \quad \forall t \in \mathbb{N}.$$

"Ehrhart polynomial"

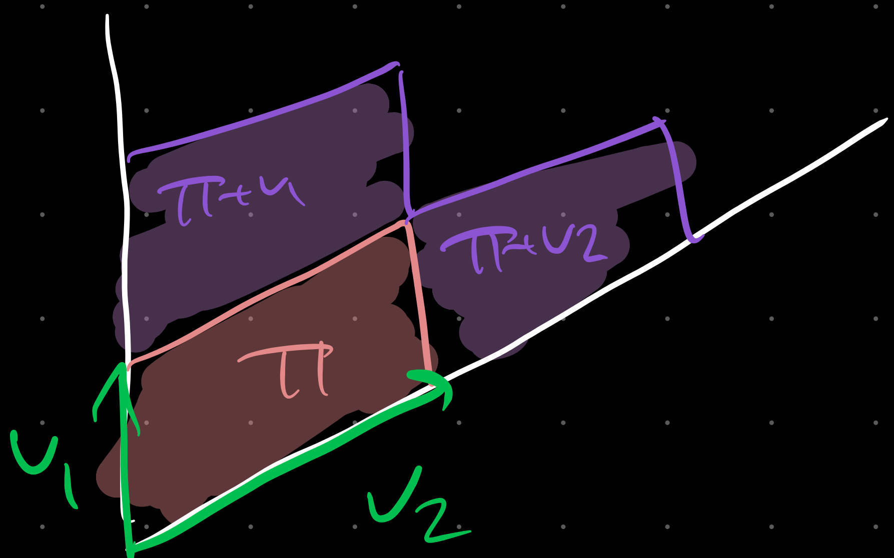
Technical lemma 2: Yes!

Idea:

$$\sigma_C(1,\dots,1,z) = \frac{\sigma_\Pi(1,\dots,1,z)}{(1-z)^{d+1}}$$

polynomial!

Idea:



$\exists$ polytope $\Pi \subseteq \mathbb{R}^{d+1}$ s.t.

$\forall x \in C$, $x = x_1 + x_2$

$x_1 \in \Pi$

$x_2 \in \mathbb{Z}^{d+1}$